Day 25 - Existence of Stiefel Whitney classes

Last time: B closed smooth manifold

$\xi : E \to B$ rank n vector bundle

All (co)homology with $\mathbb{Z}_2$-coeffs

$E_0 = E \setminus Z(B)$, non-zero vectors

$F = \pi^{-1}(b)$, generic fiber

$F_0 = F \setminus \{(b, \vec{0})\}$

$\tilde{E} = \{(b,v) \mid b \in B,\ v \cdot v \leq 1\}$

$E_1 = \partial \tilde{E} = \{(b,v) \mid v \cdot v = 1\}$, sphere bundle

$$\Rightarrow \quad H^i(F, F_0) \cong \begin{cases} \mathbb{Z}_2 & i = n \\ 0 & i \neq n \end{cases}$$

$$H^i(E, E_0) \cong \begin{cases} 0 & i < n \\ \mathbb{Z}_2 & i = n \\ H^{i-n}(B) & i > n. \end{cases}$$

More generally we have (will skip to save time):

Thom isomorphism theorem: There is a unique $u \in H^n(E, E_0) \cong \mathbb{Z}_2$ whose restriction to $H^n(F, F_0)$ is non-zero, for every fiber F.

Moreover, $\forall j$.

$$H^i(E) \xrightarrow{\cong} H^{i+n}(E, E_0).$$
$$y \longmapsto y \cup u.$$

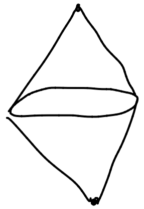
Def: The Thom isomorphism is

$$H^i(B) \xrightarrow{\varphi} H^{i+n}(E, E_0)$$
$$x \longmapsto \pi^*(x) \cup u$$

We will need an operation called Steenrod squares (see Hatcher, section 4L):

If X is a top space, $S(X)$ = suspension of X

$$S(X) = X \times I \Big/ \begin{smallmatrix} (x,0)\sim(y,0) \\ (x,1)\sim(y,1) \end{smallmatrix} \quad \forall\, x,y.$$



Ex: $S^{n+1} = S(S^n)$

<u>Thm</u>: There is an isomorphism

$$\sigma : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+1}(\Sigma X;\mathbb{Z}_2) \quad \forall n$$

(work with all coeffs, using Mayer-Vietoris).

Need $\mathbb{Z}_2$ here:

<u>Def</u>: The Steenrod squares $Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$ $\forall n$ that satisfy:

(1) $Sq^i(f^*(a)) = f^*(Sq^i(a))$ for $f: X \to Y$

(2) $Sq^i(a+b) = Sq^i(a) + Sq^i(b)$

(3) $Sq^i(a \cup b) = \sum_j Sq^j(a) \cup Sq^{i-j}(b)$

(4) $Sq^i(\sigma(a)) = \sigma(Sq^i(a))$ where σ = suspension iso.

(5) $Sq^n(a) = a \cup a$, $Sq^i(a) = 0$ for $i > n$.

(6) $Sq^0 = id$.

There are called squares since they are gen. of a square with cup products.

Squares also work in relative cohomology, like $\cup$

We will show the existence of these next time. Using these,

For a rank n v.b. $\xi : E \xrightarrow{\pi} B$, define

$$W_i(\xi) = \varphi^{-1}(Sq^i(\varphi(1)))$$

$$\begin{array}{ccccccc} H^0(B) & \xrightarrow{\varphi} & H^n(E,E^0) & \xrightarrow{Sq^i} & H^{n+i}(E,E^0) & \xrightarrow{\varphi^{-1}} & H^i(B) \\ 1 & & u & & Sq^i(u) & & W_i(\xi) \end{array}$$

Hence $W_i(\xi)$ is the unique element that goes to $Sq^i(u)$ under the Thom isomorphism.

e.g. $Sq^n(u) = u \cup u$ so $W_i(\xi) = \varphi^{-1}(u \cup u)$

<u>Verify the axioms</u>

(A1) $W_0(\xi) = \varphi^{-1}(id(\varphi(1))) = 1$ since $Sq^0 = id$

For $i > n$, $Sq^i = 0 \Rightarrow W_i(\xi) = 0$

(A2) If $f : B \to B'$ is covered by a bundle map $\xi \to \xi'$.

Then $\exists\ \bar{f} : E \to E'$ and hence $f : (E, E_0) \to (E', E_0')$

Moreover $\bar{f}^*(u') = u$, where u, u' are Thom classes.

$$\begin{array}{ccc} H^i(B') & \xrightarrow{-\cup u'} & H^{n+i}(E', E_0') \\ \downarrow f^* & & \downarrow \bar{f}^* \\ H^i(B) & \xrightarrow{-\cup u} & H^{n+i}(E, E_0) \end{array}$$

Commutes since for $y' \in H^i(E')$,

$$\bar{f}^*(y' \cup u') = \bar{f}^*(y') \cup \bar{f}^*(u') = \bar{f}^*(y') \cup u.$$

Moreover, since f is covered by a bundle map $\bar{f}$,

$$\begin{array}{ccc} H^i(B') & \xrightarrow{(\pi')^*} & H^i(E') \\ \downarrow f^* & & \downarrow \bar{f}^* \\ H^i(B) & \xrightarrow{\pi^*} & H^i(E) \end{array}$$

Hence $\bar{f}^* \circ \varphi' = \varphi \circ f^*$.

So
$$\begin{aligned} \varphi \circ f^*(w_i(\xi')) &= \varphi \circ f^*((\varphi')^{-1}(Sq^i(u'))) \\ &= \bar{f}^* \circ \varphi' \circ (\varphi')^{-1} \circ Sq^i(u') \\ &= \bar{f}^*(Sq^i(u')) \\ &= Sq^i(\bar{f}^*(u')) \\ &= Sq^i(u) \end{aligned}$$

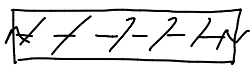
$$\Rightarrow f^*(w_i(\xi')) = \varphi^{-1} Sq^i(u) = w_i(\xi).$$

<u>Skip Axiom 3</u> (read in book).

<u>Axiom 4:</u> $w_1(\gamma_1^1) \neq 0$.

γ_1^1 is the Mobius bundle over S^1.

Consider the vectors of length ≤ 1 in $E(\gamma_1^1)$: M.



S^1 M

$$H^i(E,E_0) \cong H^i(M,\partial M)$$

Moreover $\partial M = S^1$ which we can cap off by a disk and get $\mathbb{RP}^2$.

$$M \cup_{S^1} D^2 = \mathbb{RP}^2 \qquad j: M \longrightarrow \mathbb{RP}^2$$

By excision

$$H^i(M,\partial M) \cong H^i(\mathbb{RP}^2, D^2)$$

Consider LES of pair $(\mathbb{RP}^2, D^2)$

$$\underbrace{\cancel{\tilde{H}^i(D^2)}}_{0} \longleftarrow \tilde{H}^i(\mathbb{RP}^2) \underset{\cong}{\overset{i^*}{\longleftarrow}} H^i(\mathbb{RP}^2, D^2) \longleftarrow \cancel{\tilde{H}^{i-1}(D^2)}$$

(all pres. $\cup$)

$$\underset{u}{H^1(E,E_0)} \xrightarrow{\cong} H^1(M,\partial M) \xrightarrow{\cong} H^1(\mathbb{RP}^2, D^2) \xrightarrow{\cong} H^1(\mathbb{RP}^2)$$

Since $\cong$ $u \longmapsto$ non-zero, hence $a \in H^1(\mathbb{RP}^2; \mathbb{Z}_2)$, $a \neq 0$.

Since $a \cup a \neq 0$ in $H^2(\mathbb{RP}^2; \mathbb{Z}_2)$, $u \cup u \neq 0$ (since $\cong$ also holds in $i=2$)

$$Sq^1(u) = u \cup u \neq 0$$

$$\Rightarrow w_1(\gamma_1^1) = \varphi^{-1}(Sq^1(u)) \neq 0.$$

Just need to sketch existence of Sq^i.