Day 26 - Existence of Steenrod squares

Let G be an abelian group, and $n \geq 1$.

Recall $K(G,n)$, an Eilenberg-Maclane space, is a top. space Y with $\pi_n(Y) \cong G$ and $\pi_i(Y) \cong 0$ for $i \neq n$. It is well-defined up to homotopy equivalence (among CW-complexes). Moreover, we can choose Y s.t.

$$Y = \Big(\bigvee_{\gamma \in \Lambda} S^n\Big) \cup (p\text{-cells for } p \geq n+1).$$

Thus the Hurewicz map $H: G = \pi_n(Y) \xrightarrow{\cong} H_n(Y;\mathbb{Z})$ is an isomorphism, and so is the Kronecker map

$$K: H^n(Y;G) \longrightarrow \mathrm{Hom}(H_n(Y;\mathbb{Z}), G) = \mathrm{Hom}(\pi_n(Y), G).$$

Thus, there is a fundamental class $\Lambda \in H^n(Y;G) \leftarrow$ cellular cohom. which is the cocycle that sends an n-cell, representing a n-sphere in Y to the element in $\pi_n(Y) = G$.

<u>Thm</u>: $[X, K(G,n)] \xrightarrow{\cong} H^n(X;G)$ for any CW-complex X

$$f \longmapsto f^*(\Lambda) \qquad \Lambda \text{ fund. class in } H^n(K(G,n),G).$$

[Sketch proof of $G = \mathbb{Z}$, $n=2$ if time.].

Again, we'll be interested in cohomology with $\mathbb{Z}_2$-coeffs.

Recall, there is a natural cross product

$$\times : H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X)$$

in cellular cohomology given by

$$(\alpha \times \beta)(c) = \begin{cases} \alpha(e^i)\beta(e^j) & \text{if } c = e^i \times e^j,\ i \text{ and } j \text{ cells in } X \\ 0 & \text{otherwise} \end{cases}$$

(Can also define in singular coh).

Let $\Delta : X \longrightarrow X \times X$ be the diagonal embedding.

<u>Thm</u> (see Munkres) For $\alpha \in H^p(X)$, $\beta \in H^q(X)$,

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

<u>Note</u>: For a CW-complex Δ is not cellular, so the RHS only makes sense in singular (as well as LHS).

(true for all ring coeffs).

For $\alpha \in H^n(X;\mathbb{Z}_2)$, $\alpha \times \alpha \in H^{2n}(X \times X;\mathbb{Z}_2) \cong [X \times X, \mathbb{Z}_2)$

so we can represent $\alpha \times \alpha$ by a map $f_\alpha : X \times X \longrightarrow K(\mathbb{Z}_2, 2n)$

Moreover, $\alpha \cup \alpha \in H^{2n}(X;\mathbb{Z}_2)$ can be represented by a map $g_\alpha : X \longrightarrow K(\mathbb{Z}_2, \mathbb{Z})$. Since $(f_\alpha \circ \Delta)^* = \Delta^* \circ f_\alpha^*$, we have.

$$\begin{tikzcd} X \times X \arrow[r, "f_\alpha"] & K(\mathbb{Z}_2, 2n) \\ X \arrow[u, "\Delta"] \arrow[ur, "g_\alpha"'] & \end{tikzcd}$$

Since $(f_\alpha \circ \Delta)^*(\Lambda) = \Delta^*(\alpha \times \alpha) = \alpha \cup \alpha = g_\alpha^*(\Lambda)$,

so $f_\alpha \circ \Delta$ and g_α are homotopic.

With $\mathbb{Z}_2$-coeffs, $\times$ and $\cup$ are commutative, so if
$T : X \times X \longrightarrow X \times X$ then $T^*(\alpha \times \alpha) = \alpha \times \alpha$.
$(x,y) \longmapsto (y,x)$

But $T^*(\alpha \times \alpha)$ is represented by map $f_\alpha \circ T$ so f_α and $f_\alpha \circ T$ are homotopic. Let $F : X \times X \times I \longrightarrow K(\mathbb{Z}_2, 2n)$ be the homotopy $(F(-,0) = f_\alpha,\ F(-,1) = f_\alpha \circ T)$.

Since $T^2 = id$, we can compose F & $T \times id$ $\quad (T \times id : X \times X \times I \circlearrowleft)$ to get a homotopy $F \circ (T \times id)$ with $(F \circ T \times id)(-,0) = f_\alpha \circ T$ and $(F \circ (T \circ id))(-,1) = f_\alpha \circ T^2 = f_\alpha$.

Composing these gives a loop
$$S^1 \times X \times X \longrightarrow K(\mathbb{Z}_2, 2n),$$
representing an element in $H^{2n}(S^1 \times X \times X ; \mathbb{Z}_2)$.

It turns out that this map will be null-homotopic, hence this map extends to
$$D^2 \times X \times X \longrightarrow K(\mathbb{Z}_2, 2n).$$
Again, compose this T to get map
$$S^2 \times X \times X \longrightarrow K(\mathbb{Z}_2, 2n)$$
which is null-homotopic.

$\vdots$

Eventually, we get a map
$$S^\infty \times X \times X \longrightarrow K(\mathbb{Z}_2, 2n)$$

s.t. (s,x,y) and $(-s,y,x)$ go to the same point in $K(\mathbb{Z}_2,2n)$. Thus

$$\begin{array}{ccccc}
S^\infty \times X & \xrightarrow{Id\times\Delta} & S^\infty \times (X\times X) & \longrightarrow & K(\mathbb{Z}_2, 2n) \\
(s,x) & \longmapsto & (s,x,x) & \longrightarrow & \text{same.} \\
(-s,x) & \longmapsto & (-s,x,x) & \nearrow &
\end{array}$$

$\rightsquigarrow \mathbb{R}P^\infty \times X \longrightarrow K(\mathbb{Z}_2,2n)$. This represents an element in $H^{2n}(\mathbb{R}P^\infty \times X;\mathbb{Z}_2)$

Kunneth Thm says that every element in $H^{2n}(\mathbb{R}P^\infty \times X)$ is of form $\sum_i [\gamma]\times[\alpha_i]$ where $[\gamma]\in H^{2n-i}(\mathbb{R}P^\infty;\mathbb{Z}_2)$

$[\alpha_i]\in H^i(X;\mathbb{Z}_2)$

But $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)\cong \mathbb{Z}_2[w]$ $w\neq 0$ in $H^1(\mathbb{R}P^\infty,\mathbb{Z}_2)=\mathbb{Z}_2$.

So each element is

$$\sum_i w^{2n-i}\times \alpha_i \quad \text{for some } \alpha_i.$$

Moreover, α_i is unique. Define

$$Sq^i(\alpha)=\alpha_{n+i}.$$