Day 27 - Applications of Stiefel-Whitney Classes

Recall we have the following bijections for rank 1 v.b.s

$$\mathrm{Vect}_1^{\mathbb{R}}(B) \longleftrightarrow [B, G_1(\mathbb{R}^\infty)] = [B, \mathbb{R}P^\infty] \longleftrightarrow H^1(B;\mathbb{Z}_2)$$

$$\mathrm{Vect}_1^{\mathbb{R}}(B) = \left\{\begin{matrix}\text{rank 1 v.b.}\\ \text{over } B\end{matrix}\right\}$$

Note: $[B, \mathbb{R}P^\infty] \longrightarrow H^1(B;\mathbb{Z}_2)$

$f \longmapsto f^*(\alpha) \qquad 0 \neq \alpha \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2.$

and we have

$$[B, \mathbb{R}P^\infty] \to \mathrm{Hom}(\pi_1(B), \pi_1(\mathbb{R}P^\infty)) = \mathrm{Hom}(\pi_1(B), \mathbb{Z}_2) = \mathrm{Hom}(H_1(B), \mathbb{Z}_2) \xleftarrow[\cong]{\kappa} H^1(B;\mathbb{Z}_2)$$

$$f \longmapsto f_* \longleftarrow f^*(\alpha)$$

Prop: $\xi \longrightarrow w_1(\xi)$ under this bijections (as asserted before). Hence for rank 1 $\mathbb{R}$-vector bundles,

$$\xi \cong \xi' \iff w_1(\xi) = w_1(\xi').$$

Pf: Given ξ, choose $f: B \to \mathbb{R}P^\infty$, a classifying map hence $w_1(\xi) = f^*(w_1(\gamma^1))$. But $w_1(\gamma^1) = a$ the gen. of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$.

$\boxtimes$

Prop: If ξ is a rank n vector bundle with a non-zero section then $w_n(\xi) = 0$. More generally, if ξ admits k everywhere lin. independent sections then

$$w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0.$$

Pf: Suppose ξ has k lin. ind. sections. Put a Riemannian metric on ξ. Then $\xi \cong \varepsilon^k \oplus (\varepsilon^k)^\perp$ with $(\varepsilon^k)^\perp$ of dim.

$n-k$. Thus $w_i(\xi) = w_i((\varepsilon^k)^\perp) = 0$ for $i > n-k$.

In class exercise: prove that the two definitions of the Stiefel-Whitney classes coincide. See the handout for solution

[See MS for computation of $w(\mathbb{R}P^n)$, Thm 4.5]

Thm 4.5 of MS: $T\mathbb{R}P^n \oplus \varepsilon^1 \cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1$. Hence

$$w(\mathbb{R}P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \cdots + \binom{n+1}{n} a^n. \quad (\text{mod } 2)$$

Ex: coeffs of $w(\mathbb{R}P^n)$:

$n=1$: 1 0
$n=2$: 1 1 1
$n=3$: 1 0 0 0
$n=4$: 1 1 0 0 1
$n=5$: 1 0 1 0 1 0
$n=6$: 1 1 1 1 1 1 1
$n=7$: 1 0 0 0 0 0 0 0
$\vdots$

Cor (Stiefel): $w(\mathbb{R}P^n) = 1 \iff n+1$ is a power of 2

Hence only possible parallelizable $\mathbb{R}P^n$ are $\mathbb{R}P^1 = S^1$, $\mathbb{R}P^3$, $\mathbb{R}P^7$, $\mathbb{R}P^{15}$, ...

Remark: It's known that $\mathbb{R}P^1, \mathbb{R}P^3, \mathbb{R}P^7$ are parallelizable but rest are not (harder).