Day 31 - Euler class

For ξ an oriented bundle, we have the Thom isomorphism with $\mathbb{Z}$-coeffs.

Recall if ξ is oriented then each fiber F has a basis $v_1, \dots, v_n$ that gives the orientation that continuously varies over B. Then $H_n(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$ and has a preferred generator u_F. Moreover $H^n(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$ has a preferred generator v_F, $\langle v_F, u_F \rangle = 1$.

<u>Theorem</u>: Let ξ be an oriented n-plane bundle $\xi : E \to B$, with F a fiber. Then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$ and $H^n(E, E_0; \mathbb{Z})$ contains a unique u s.t.

① $u|_{(F,F_0)} = u_F$.

②
$$\begin{aligned} H^k(E;\mathbb{Z}) &\xrightarrow{\cong} H^{n+k}(E, E_0; \mathbb{Z}) \text{ is an } \cong \ \forall k. \\ y &\longmapsto y \cup u \end{aligned}$$

In particular since $H^k(E) \cong H^k(B)$, we have the Thom isomorphism

$$\begin{aligned} \phi : H^k(B;\mathbb{Z}) &\longrightarrow H^{k+n}(E, E_0; \mathbb{Z}) \\ x &\longmapsto \pi^*(x) \cup u \end{aligned}$$

Consider $(E, \phi) \to (E, E_0) \rightsquigarrow H^*(E, E_0) \longrightarrow H^*(E)$

$$y \longmapsto y|_E$$

<u>Def</u>: The Euler class of an oriented bundle ξ is

$$e(\xi) = (\pi^*)^{-1}(u|_E) \in H^n(B; \mathbb{Z}).$$

$$\begin{array}{ccccc} H^n(E,E_0) & \longrightarrow & H^n(E) & \xrightarrow[\cong]{(\pi^*)^{-1}} & H^n(B) \\ u & \longmapsto & u|_E & \longrightarrow & (\pi^*)^{-1}(u|_E) = e(\xi). \end{array}$$

Properties:

① $f : B \longrightarrow B'$ then $e(f^*(\xi)) = f^*(e(\xi))$

$\Rightarrow e(\text{trivial bundle}) = 0 \quad (n>0).$

② If the orientation of ξ is reversed then $e(\xi)$ change by a sign.

③ If n is odd then $e(\xi)+e(\xi)=0$.

Pf: Consider Thom iso

$$\begin{array}{ccccc} H^n(B) & \xrightarrow{\pi^*} & H^n(E) & \longrightarrow & H^{n+n}(E,E_0) \\ e(\xi) & \longmapsto & \pi^*(e(\xi)) & \longmapsto & \pi^*(e(\xi)) \cup u = u \cup u \\ & & & & \overset{\shortparallel}{u|_E} \end{array}$$

So $e(\xi) = \phi^{-1}(u \cup u)$, where ϕ is the Thom iso.

But $a \cup b = (-1)^{\dim a \cdot \dim b}\, b \cup a \Rightarrow u \cup u$ has order since $n =$ odd. Then $e(\xi)$ has order 2.

Prop: The homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ induces

$$\begin{array}{ccc} H^n(B;\mathbb{Z}) & \longrightarrow & H^n(B;\mathbb{Z}_2) \\ e(\xi) & \longmapsto & w_n(\xi). \end{array}$$

Pf: Recall $w_n(\xi) = \phi^{-1}(Sq^n(u))$ and $Sq^n(u) = u \cup u$. where $u \in H^n(E,E_0;\mathbb{Z}_2) \cong \mathbb{Z}$ and ϕ is the Thom iso with $\mathbb{Z}_2$-coeffs. In addition the integer thom class goes to the $\mathbb{Z}_2$-Thom class. $\boxed{\xi}$

<u>Prop:</u> $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$
$e(\xi \times \xi') = e(\xi) \times e(\xi')$.

Not as useful as before!

- $w(\xi)$ is a unit in ring $H^{\Pi}(B;\mathbb{Z}_2)$ so can solve for $w(\xi')$ as a function of $w(\xi)$ and $w(\xi\oplus\xi')$.

However $e(\xi)$ not unit in integral cohomology, could be zero in fact!

<u>Cor:</u> If $2e(\xi)\neq 0$, then ξ can't split as sum of two odd dim manifolds.

<u>Ex:</u> Let M be a smooth compact mfld with $w_1(M)=0$ (i.e TM is orientable). Choose an orientation. If TM has an orientable odd dim'l subbundle ξ then

$$e(M) = \underset{\substack{\uparrow \\ \text{order } 2}}{e(\xi)} \oplus e(\xi^\perp)$$

$\Rightarrow 2e(M) = 0$ in $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$ $\Rightarrow$ $\boxed{e(M)=0}$

In particular, if M is orientable then every nowhere zero v.f. gives a trivial sub-bundle of dim 1.

$$e(M) = \underset{\substack{\shortparallel \\ 0}}{e(\xi)} \cup e(\xi^\perp) = 0$$

Prop: If the oriented v.b. ξ admits a nowhere zero section then $e(\xi) = 0$.

Thm: $\langle e(TM), [M] \rangle = \chi(M)$. for a closed orientable manifold.

Note: We have

$$H^n(M;\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_n(M), \mathbb{Z})$$

hence $e(TM) = 0 \Leftrightarrow \chi(M) = 0$.