First Stiefel-Whitney class

We will show that the two definitions of the first Stiefel-Whitney class coincide.

Let $\xi : E \xrightarrow{\pi} B$ be a rank n vector bundle over B, with B paracompact.

Def: Define $\varphi_\xi \in \mathrm{Hom}(\pi_1(B), \mathbb{Z}_2)$ as follows. For $c : S^1 \longrightarrow B$, define

$$\varphi_\xi([c]) = \begin{cases} 0 & \text{if } c^*(\xi) \text{ is trivial} \\ 1 & \text{if } c^*(\xi) \text{ is non-trivial} \end{cases}$$

φ_ξ is well-defined since if c_1, c_2 are homotopic then $c_1^*(\xi) \cong c_2^*(\xi)$. Moreover, there are only two bundles over S^1, ε^n and $MB \oplus \varepsilon^{n-1}$, by the clutching construction since $\pi_0(GL_n(\mathbb{R}))$ has two elements.

Recall $\kappa : H^1(B; \mathbb{Z}_2) \longrightarrow \mathrm{Hom}(H_1(B), \mathbb{Z}_2)$ is an $\cong$.

Let $\tilde{A} : \mathrm{Hom}(H_1(B), \mathbb{Z}_2) \longrightarrow \mathrm{Hom}(\pi_1(B), \mathbb{Z}_2)$ be the Hom dual of the abelianization map A.

Since $\mathbb{Z}$ is abelian, $\tilde{A}$ is an $\cong$. Hence we have an iso

$$H^1(B; \mathbb{Z}_2) \xrightarrow{\kappa} \mathrm{Hom}(H_1(B), \mathbb{Z}_2) \xrightarrow{\tilde{A}} \mathrm{Hom}(\pi_1(B), \mathbb{Z}_2)$$

Define $\boxed{\tilde{w}_1(\xi) := (\tilde{A} \circ \kappa)^{-1}(\varphi_\xi).}$

Def 2: Choose a classifying map for ξ, $f: B \longrightarrow G_n(\mathbb{R}^\infty)$. Define

$$\boxed{w_1(\xi) := f^*(w_1)}$$

where $w_1 = w_1(\gamma^n) \in H^1(G_n(\mathbb{R}^\infty); \mathbb{Z}_2)$.

Theorem: $w_1(\xi) = \tilde{w}_1(\xi)$.

Proof: Let $f: B \longrightarrow G_n$ be a classifying map for ξ. So $\xi \cong f^*(\gamma^n)$ and we have the bundle map:

$$\begin{tikzcd} f^*(\gamma^n) \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\ B \arrow[r, "f"] & G_n \end{tikzcd}$$

By definition, $w_1(\xi) = f^*(w_1(\gamma^n))$. It suffices to show that $\tilde{A} \circ \mathcal{K}(f^*(w_1(\gamma^n))) = \varphi_\xi$ as defined above. Let $c: S^1 \longrightarrow B$ then $c^*(\xi) \cong c^*(f^*(\xi)) \cong (f \circ c)^*(\xi)$ as rank n vector bundles over S^1. But

$$\mathrm{Vect}_n^{\mathbb{R}}(S^1) \xleftrightarrow{1:1} [S^1, G_n]$$
$$(f \circ c)^*(\xi) \longleftarrow\!\shortmid \; f \circ c$$

So $c^*(\xi) = (f \circ c)^*(\xi)$ is trivial $\Leftrightarrow$ $f \circ c$ is nullhomotopic $\Leftrightarrow$ $[f \circ c] = 1$ in $\pi_1(G_n)$

$$\Leftrightarrow f_*([c]) = 1 \text{ in } \pi_1(G_n)$$

We claim that $\pi_1(G_n) \cong \mathbb{Z}_2$. To see this note that G_n has a CW-complex with a single 1-cell and at most two 2-cells and $G_1 \subseteq G_n$ is a subcomplex, from the cell-structure of G_n. $G_1 = \mathbb{RP}^\infty$ has a single 1-cell and a single 2-cell and the attaching map is given by $z \to z^2$ map. In particular, $\pi_1(G_1) = \mathbb{Z}_2$. When $n \geq 2$, there is another two cell attached which means that $\pi_1(G_n) \cong \pi_1(G_1)/e$ for some $e \in \pi_1(G_1)$ so $\pi_1(G_n) = 1$ or $\mathbb{Z}_2$. But we have

$$\pi_1(G_n) \twoheadrightarrow [S^1, G_n] \overset{1\text{-}1}{\longleftrightarrow} \mathrm{Vect}_n^{\mathbb{R}}(S^1) = \{\varepsilon^n, MB \oplus \varepsilon^{n-1}\}$$

so $\pi_1(G_n) \neq 1$ hence $\pi_1(G_n) \cong \mathbb{Z}_2$. Thus

$$\varphi_\xi([c]) = \begin{cases} 1 & \text{if } f_*([c]) \neq 1 \\ 0 & \text{if } f_*([c]) = 0 \end{cases}.$$

We claim that $\tilde{A} \circ \mathcal{K} \circ f^*(w_1(\gamma^n)) = \varphi_\xi$. To see this note that the following diagram commutes

$$\begin{tikzcd}
H^1(B;\mathbb{Z}_2) \arrow[r, "\mathcal{K}", "\cong"'] & \mathrm{Hom}(H_1(B),\mathbb{Z}_2) \arrow[r, "\tilde{A}", "\cong"'] & \mathrm{Hom}(\pi_1(B),\mathbb{Z}_2) \\
H^1(G_n;\mathbb{Z}_2) \arrow[r, "\mathcal{K}", "\cong"'] \arrow[u, "f^*"] & \mathrm{Hom}(H_1(G_n),\mathbb{Z}_2) \arrow[r, "\tilde{A}", "\cong"'] \arrow[u, "\tilde{f}_*"] & \mathrm{Hom}(\pi_1(G_n),\mathbb{Z}_2) \arrow[u, "\tilde{f}_*"]
\end{tikzcd}$$

Hence $\tilde{A} \circ \mathcal{K} \circ f^*(w_1(\gamma^n)) = \tilde{f}_*(\tilde{A}(\mathcal{K}(w^1(\gamma^n))))$. Let $\alpha \in \pi_1(G_n)$ be the generator of $\pi_1(G_n) \cong \mathbb{Z}_2$ then $\tilde{A}(\mathcal{K}(w^1(\gamma^n)))(\alpha) = 1$. For $c: S^1 \to B$, $f_*([c]) = m\alpha$ for $m = 0$ or 1.

Hence,
$$
\begin{aligned}
f_*(\tilde{A}(\kappa(w_1(\gamma^n)))([c]) &= \tilde{A}(\kappa(w_1(\gamma^n)))(f_*([c])) \\
&= \tilde{A}(\kappa(w_1(\gamma^n)))(m\alpha) \\
&= m
\end{aligned}
$$

So $f_*(\tilde{A}(\kappa(w_1(\gamma^n))) = \varphi_\xi$.

$\blacksquare$

<u>Corollary</u>: Let ξ be a rank n vector bundle over B, with B paracompact. Choose a classifying map $f : B \to G_n$ for ξ and identify $\mathbb{Z}_2$ with $\pi_1(G_n)$. Then

$$
\begin{array}{ccc}
H^1(B;\mathbb{Z}_2) & \xrightarrow{\tilde{A} \circ \kappa} & \mathrm{Hom}(\pi_1(B), \pi_1(G_n)) \\
w_1(\xi) & \longmapsto & f_*
\end{array}
$$