Motivation for L^2-Invariants

Let X be a connected CW complex of finite type (i.e. each skeleton is finite).
recall that
$$\beta_p(X) = \operatorname{rank} H_p(X) < \infty.$$
This may not give you much information and it is often useful to pass to a covering space to get more info.
For example, $H_p(S^3 \setminus \text{Knot}) \cong H_p(S^1)$ $\forall\, p$. However if we consider the Alexander module we can distinguish knots as follows:
Let K be a knot in S^3, $X_K = S^3 \setminus K$, and $\tilde{X}_K$ be the infinite cyclic cover corresponding to
$$\pi_1(X_K) \twoheadrightarrow H_1(X_K) \cong \mathbb{Z} = \langle t \rangle.$$
↖ meridian

Define the Alexander module to be
$$\mathcal{A}(K) := H_1(\hat{X}_K).$$
This is a left $\mathbb{Z}[t, t^{-1}]$-module since $H_1(X_K) = \operatorname{Deck}(\tilde{X}_K)$.
and one can show that it is a torsion module.
$\mathbb{Z}[t, t^{-1}]$ is not quite a PID but $\mathbb{Q}[t, t^{-1}]$ is

and $A(K)$ is $\mathbb{Z}$ torsion free so

$$A(K) \hookrightarrow A(K) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \underset{\substack{\uparrow \\ \mathbb{Q}[t^{\pm 1}]\text{-module.}}}{H_1(\tilde{X}_K;\mathbb{Q})}$$

and $H_1(\tilde{X}_K;\mathbb{Q}) \cong \mathbb{Q}[t^{\pm 1}]/p_1(t) \oplus \cdots \oplus \mathbb{Q}[t^{\pm 1}]/p_s(t)$

The <u>Alexander polynomial of K</u> is $\Delta_K(t) = p_1(t) \cdots p_s(t)$.

Alternatively $\Lambda = \mathbb{Z}[t^{\pm 1}]$ is Noetherian and $A(K)$ is a f.g. Λ-module so $A(K)$ has a presentation

$$\Lambda^n \xrightarrow{T} \Lambda^m \longrightarrow A(K)$$

In fact, can assume that $m = n$. One can compute this using group theory (from $\pi_1(X)$) or topologically (using a Seifert surface). The ideal gen by $m \times m$ minors is called the <u>Alexander ideal</u> $I_0(K)$.

For knots, $I_0(K)$ is principal and $\Delta_K(t)$ is the generator.

<u>Ex:</u> $A(\text{unknot}) = 0 \Rightarrow \Delta_{\text{unknot}}(t) = 1$.

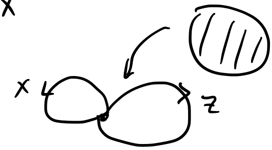
<u>Ex:</u> T = trefoil

$$G = \pi_1(S^3 - T) = \langle x, y \mid xyx = yxy \rangle \leftarrow B_3 \quad \text{let } z = yx^{-1}$$

$$= \langle x, z \mid xzx = zx^2z \rangle$$

$$= \langle x, z \mid zx^2zx^{-1}z^{-1}x^{-1} \rangle \leftarrow \text{note } z = 0 \text{ in } H_1(S^3 - T).$$

Let X = 2-complex



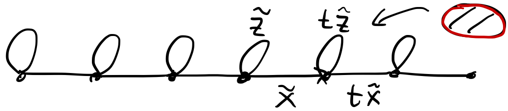
add 2-cell corr. to relation $z x^2 z x^{-1} z^{-1} x^{-1}$

Then $\pi_1(X) \cong \pi_1(S^3 \setminus T) \xrightarrow{Ab} \mathbb{Z} = \langle t \rangle \leftarrow H_1(X) \cong H_1(S^3 \setminus T)$

$\Rightarrow \pi_1(\tilde{X}) \cong \ker(ab) \cong \pi_1(\widetilde{S^3 \setminus T})$

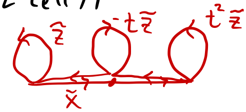
$\Rightarrow H_1(\tilde{X}) \cong ab(\ker(Ab)) \cong H_1(\widetilde{S^3 \setminus T}) \cong A(T)$.

$\tilde{X}$:



add copies of 2-cell (one for each t^m).

Let $c = \tilde{z}$, then c generates $H_1(\tilde{X})$. There is one relation (lift of ∂(2-cell))



So

$$A(T) \cong \mathbb{Z}[t^{\pm 1}] / \langle 1 - t + t^2 \rangle \quad +$$

$$\Delta_T(t) = 1 - t + t^2.$$

$\Rightarrow T \neq$ unknot.

<u>Alternatively</u>: pick a splitting $\pi_1(X) \longrightarrow \mathbb{Z} = \langle t \rangle$, say $t \mapsto x$.
Then t acts by $t \cdot g := x g x^{-1}$ for any $g \in \pi_1(\tilde{X})$.
This is well defined on $H_1(\tilde{X})$. So relation in $\pi_1(X)$ becomes $z \cdot x^2 z x^{-2} \cdot x z^{-1} x^{-1} \xrightarrow[\text{in } H_1]{} c + t^2 c - tc$

$$= c(1 - t + t^2)$$

Remark: Easy to generalize this for CW complexes of finite type.

For general X, there is a simpler invariant.

Suppose $\beta_1(X) = m \geq 1$. Let $\tilde{X}$ be the universal abelian cover of X, i.e. cover corresponding to

$$\pi_1(X) \longrightarrow H_1(X) \longrightarrow H := H_1(X)/\text{torsion}.$$

Then $H \cong \mathbb{Z}^m = \langle x_1^{\pm 1}, \ldots, x_m^{\pm 1} \rangle$ (mult) and so $H_1(\tilde{X})$ is a left $\mathbb{Z}[H]$-module, a Noetherian ring with field of fractions $Q(H) \cong Q(x_1, \ldots, x_m) \leftarrow P/q$.

Consider $H_1(\tilde{X}) \otimes_{\mathbb{Z}H} Q(H) \cong Q(H)^{r(X)}$.

Lemma: $r(X) \leq \beta_1(X) - 1$

Rmk: For knots, $r = 0$ $\leftarrow$ have torsion Alexander modc.

Def: The Alexander nullity of X is $r(X)$.

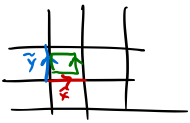
For a link, this may be nonzero.

Ex:

$L =$  $\Rightarrow \pi_1(S^3 - L) \cong \mathbb{Z} * \mathbb{Z}$

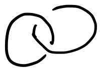
Let $W =$ 

then $H_1(\tilde{W}) \cong H_1(\widetilde{S^3 - L}) \cong A(L)$

: 

Let $c = \widetilde{xyx^{-1}y^{-1}}$

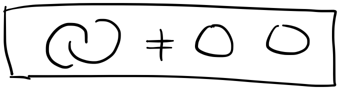
Then $A(L) \cong H_1(\tilde{W}) \cong \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \Rightarrow r(W) = r(L) = \underline{1}$.

Ex : $L =$  $\Rightarrow \pi_1(S^3 - L) = \mathbb{Z}^2$

$\Rightarrow \pi_1(\widetilde{S^3 - L}) = 0$

$\Rightarrow A(L) = H_1(\widetilde{S^3 - L}) = 0$

$\Rightarrow r(L) = 0$.



We can think of $r(X)$ as an equivariant Betti number of the cover. Note $\beta_1(\tilde{X}) = \infty \Leftarrow r(X) \geq 1$. So $r(X)$ can give us info where $\beta_1(\tilde{X})$ does not.

Other ∞ covers?

Consider X_{univ}, universal cover of X.

$\rightsquigarrow \mathrm{Deck}(X_{univ} \to X) \cong \pi_1(X)$

and hence $H_1(X_{univ})$ is a left $\mathbb{Z}\pi_1(X)$-module. This is NOT a nice ring.

Ex: $W =$ 

$F = \pi_1(W) = \langle x, y \rangle$ then $H_1(\tilde{W})$ is a left $\mathbb{Z}F = \mathbb{Z}\langle x, y \rangle$ module

↑ non-comm. polys.

Not a "nice" ring, even after tensoring with $\mathbb{Q}$ or $\mathbb{C}$.

Suppose $R \hookrightarrow K$ = field. Then for any M a f.g. left R-mod. $K \otimes_R M$ is a f.g. K-module so $K \otimes_R M \cong K^r$ for some r. In particular, $K \otimes_R R^m = K^m$.

<u>Ex</u>: Let $R = \mathbb{C}\langle x, y \rangle$ then

$$0 \to \mathbb{C}\langle x, y\rangle^2 \xrightarrow[\substack{(x-1,\, y-1) \\ \text{injective.}}]{} \mathbb{C}\langle x, y\rangle \to \underset{\substack{\shortparallel \\ \text{quotient.}}}{M} \to 0 \quad \text{exact}$$

Suppose $\mathbb{C}\langle x, y\rangle \hookrightarrow K$. Then

$$0 \to K^2 \longrightarrow K \longrightarrow K \otimes M \longrightarrow 0$$

$\Rightarrow$ rank $K \otimes M$ cannot be additive!

In fact, can only construct a right ring of fractions if R is an Ore domain.

So we move to Von Neumann dimension.