Bounded operators

<u>Def</u>: If H is a Hilbert space,

$$B(H) = \{\text{bounded linear operators } H \circlearrowleft\}.$$

i.e. for $A \in B(H)$, $\exists$ number $M > 0$ s.t. $\|Ax\| \leq M\|x\|$ $\forall x \in H$.

* <u>Prop</u> (std in any FA book):

$A \in B(H) \iff A$ is continuous.

Recall that for any $n \times n$ matrix over $\mathbb{C}$, we have that

$$\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, \overline{A}^T\vec{y}\rangle$$

We call $\overline{A}^T$ the adjoint of A, and write it $A^* := \overline{A}^T$
(conjugate transpose)

<u>Def</u>: For $L \in B(H)$, the <u>adjoint</u> of A is the unique operator A^* s.t. $\langle Af, g\rangle = \langle f, Ag\rangle$ $\forall f, g \in H$.

[Can define for $B(\mathcal{H}, \mathcal{K})$ = bounded operators $\mathcal{H} \to \mathcal{K}$].

<u>Def</u>: $A \in B(H)$,

- A is self-adjoint (Hermitian) if $A = A^*$.
- A is normal if $A^*A = AA^*$

- All the usual properties hold:

  - $(AB)^* = B^*A^*$
  - $A^{**} = A$

  $\vdots$

Can define norm on $A \in B(\mathcal{H})$.

$$\|A\| = \sup \{ \|Af\| \mid f \in \mathcal{H}, \|f\| \leq 1 \}.$$

<u>Prop</u>: (1) $A \in B(\mathcal{H})$, $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$

(2) [only true for $\mathbb{C}$ Hilbert space] A is hermitian $\Leftrightarrow$

$\langle Af, f \rangle \in \mathbb{R} \quad \forall f \in \mathcal{H}$.

Not true for $\mathbb{R}$-Hilbert space: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on $\mathbb{R}^2 \Rightarrow$

$\langle Ax, x \rangle = 0 \quad \forall x \in \mathbb{R}^2$, $A^* = A^T \neq A$.

<u>Prop</u>: If $A = A^* \Rightarrow \|A\| = \sup \{ |\langle Ah, h \rangle| \mid \|h\| = 1 \}$.

<u>Cor</u>: [$\mathcal{H}$ $\mathbb{C}$-Hilbert space]. If $A \in B(H)$ s.t. $\langle Ah, h \rangle = 0$ $\forall h \in \mathcal{H} \Rightarrow A = 0$.