Ring Theoretic properties of $N\Gamma$

Even though $\ell^2\Gamma$ is not a ring, $N\Gamma$ is a ring. and contains $\mathbb{C}\Gamma$ as a subring. We investigate its algebra.

(1) Let $H \leq G$. and $i: H \to G$ it's group homomorphism. This induces a ring hom. $i: \mathbb{C}H \to \mathbb{C}G$. which extends to a ring homomorphism

$$N(i): NH \longrightarrow NG$$

as follows: Let $g: \ell^2 H \to \ell^2 H$ be an H-equivariant bounded operator. Then

$$\mathbb{C}G \otimes_{\mathbb{C}H} \ell^2 H \subseteq \ell^2\Gamma$$

is a dense G-invariant subspace of $\ell^2 G$ and

$$id \otimes g: \mathbb{C}G \otimes_{\mathbb{C}H} \ell^2 H \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}H} \ell^2 H$$

is a G-equivariant linear map which is bounded wrt to norm from $\ell^2 G$. Hence it induces a G-equiv. bounded operator $\ell^2 G \longrightarrow \ell^2 G$, which we call $N(i)(g)$.

(2) $N\Gamma$ has <u>no</u> non-trivial 0-divisors $\Leftrightarrow \Gamma = 0$

(3) $N\Gamma$ is Noetherian $\Leftrightarrow G =$ finite

Yet it is still semiheriditary:

<u>Def</u>: A ring is <u>semiheriditary</u> if every f.g. submodule of a projective module is projective.

Comparison of modules and Hilbert $N\Gamma$-modules

As mentioned above, $N\Gamma$ is a ring with multiplication: for $f, g \in N\Gamma$

$$f \cdot g = f \circ g.$$

Note $L_\gamma \cdot f = f \cdot L_\gamma$ and $L_\gamma \cdot g = g \cdot L_\gamma \Rightarrow$

$$L_\gamma \cdot (f \cdot g) = (f \cdot g) \cdot L_\gamma .$$

Suppose M is a f.g. submodule of a projective module. P (in the algebraic sense) over the ring $N\Gamma$.

Let $f : N\Gamma^m \longrightarrow P$ be s.t. $Im(f) = M$. Since P is projective, P is a summand in a free module F with basis $\{b_i\}_{i \in I}$ and $f : N\Gamma^m \longrightarrow F$ with $im(f) = M$. Let $\{e_i\}$ be the standard basis for $N\Gamma^m$, $e_i = (0, \dots, \underset{\uparrow \text{unit in } N\Gamma}{e}, \dots)$.

Then the image of each e_i lives in the free mod. spanned by a finite # of $\{b_i\}$. So can assume $f : N\Gamma^m \longrightarrow N\Gamma^n$ module hom. with $im(f) = M$.

Moreover, can assume $m = n$.

Choose a matrix A s.t. $f\vec{x} = \vec{x} A$, that is f is given by ri ht mult by $A = (a_{ij}) \in M_n(N\Gamma)$ with $a_{ij} \in N\Gamma$.

$$\nu(f):(\ell^2\Gamma)^m \longrightarrow (\ell^2\Gamma)^m$$

$$(u_1,\dots,u_m) \longmapsto \Big(\sum_{i=1}^{m} \overline{a^*_{i,1}(\bar{u}_i)}, \dots, \sum_{i=1}^{m} \overline{a^*_{i,m}(\bar{u}_i)}\Big)$$

where a^*_{ij} is the adjoint of a_{ij},

$$\langle a_{ij}(x), y\rangle = \langle x, a^*_{ij}(y)\rangle \quad \forall\ x,y \in \ell^2\Gamma,$$

and $\overline{\sum_{\gamma\in\Gamma} \lambda_\gamma \gamma} = \sum_{\gamma\in\Gamma} \bar{\lambda}_\gamma \gamma$.

e.g. when $m=n=1$, $A=(a)$ $\quad a \in N\Gamma = B(\ell^2\Gamma)^\Gamma$

$$\nu(f)(u) = \overline{a^*(\bar{u})}$$

$$\nu(f)(\gamma u) = \overline{a^*(\overline{\gamma u})} = \overline{a^*(\gamma \bar{u})} = \gamma \overline{a^*(\bar{u})} = \gamma \cdot \nu(f)(u).$$

<u>Notes</u>: $\nu(\mathrm{id}) = \mathrm{id}$, ν is $\mathbb{C}$ linear, $\nu(f\cdot g) = \nu(f)\cdot\nu(g)$ and $\nu(f)^* = \nu(f^*)$.

We can also construct such a functor for $N\Gamma^m \to N\Gamma^n$ in the same way.

Conversely, if $g: \ell^2\Gamma^m \longrightarrow \ell^2\Gamma^m$ is Γ-equivariant and in $B(\ell^2\Gamma)$ then $\exists$ a <u>unique</u> $f: N\Gamma^m \longrightarrow N\Gamma^m$ that is $N\Gamma$-linear s.t.

$$g = \nu(f).$$

Moreover, the following properties hold:

(1) $N\Gamma^m \xrightarrow{f} N\Gamma^n \xrightarrow{g} N\Gamma^p$ is exact $\iff$

$\ell^2\Gamma^m \xrightarrow{v(f)} \ell^2\Gamma^n \xrightarrow{v(g)} \ell^2\Gamma^p$ is exact.

(2) $v(f^*) = v(f)^*$

Thus

<u>Define</u> a (Hilbert) <u>f.g. projectice $N\Gamma$-module</u> M to be a f.g. module over the ring $N\Gamma$ s.t. $M = \mathrm{im}(p)$, $p : N\Gamma^m \longrightarrow N\Gamma^m$ s.t.

- $f^2 = f$ and
- $f = f^*$.

We see this is equivalent to our previous definition, with $p = v(f)$.

<u>Def</u>: The <u>dimension</u> of a f.g. projectice $N\Gamma$-module M is

$$\dim_{N\Gamma}(M) := tr(A)$$

where A is an $M = \mathrm{im}(f : \ell^2\Gamma^m \to \ell^2\Gamma^m)$ and $f(x) = xA$ for some $A \in M_n(N\Gamma)$.

Here $tr(A) = \sum_{i=1}^{m} tr(a_{ii})$

We note that $\dim_{N\Gamma} M \in \mathbb{R}_{\geq 0}$ by same "proof".

* In fact, one can define $\dim_{N\Gamma}(M)$ for any $N\Gamma$-module (see later if we have time).

We can now prove the following thm.

<u>Theorem</u>: $N\Gamma$ is semihereditary for all Γ.

Let M be a f.g. submodule of a projective $N\Gamma$-module. We wish to show M is projective. Since a projective module is a summand of a free module, we can assume $M \subseteq N\Gamma^n$. Choose an $N\Gamma$-linear map $f: N\Gamma^m \longrightarrow N\Gamma^m$ with image M. Consider the kernel of $v(f): \ell^2\Gamma^m \longrightarrow \ell^2\Gamma^m$. It is a Γ-invt subspace of $\ell^2\Gamma^m$. Let $p: \ell^2\Gamma^m \longrightarrow \ell^2\Gamma^m$ s.t. $\mathrm{im}(p) = \ker(v(f))$ with $p = p^*$ and $p^2 = p$. Choose $g: N\Gamma^m \longrightarrow N\Gamma^m$ with

$$v(g) = p.$$

Then $v(g^2) = v(g) \cdot v(g) = p^2 = p = v(g)$ and $v(g^*) = v(g)^* = p^* = p = v(g)$ so

$$g^2 = g = g^*.$$

Since

$$\ell^2\Gamma^m \xrightarrow{p = v(g)} \ell^2\Gamma^m \xrightarrow{v(f)} \ell^2\Gamma^m$$

is exact,

$$N\Gamma^m \xrightarrow{g} N\Gamma^m \xrightarrow{f} N\Gamma^m$$

is exact. Hence $\ker(f) = \mathrm{im}(g) =$ projective.

Since $\mathrm{im}(g)$ is projective, and this is exact,

$$0 \to \mathrm{im}(g) \hookrightarrow \ell^2\Gamma^m \xrightarrow{f} \mathrm{im}(f) = M \to 0 \quad (**)$$

(**) splits, so M is projective.

Idea: One passes to Hilbert space and uses orthogonality & projections there. Hence by passing from $\mathbb{C}\Gamma$ to $N\Gamma$, we have a ring "with orthogonality".