Example: von Neumann trace for $N\mathbb{Z}^m$

Let $T^m = S^1 \times \cdots \times S^1 = \mathbb{R}^m/\mathbb{Z}^m$ be the m-torus with usual measure μ.

Let $f \in L^p(T^m)$ if $f : T^m \longrightarrow \mathbb{C}$ is s.t.

$$\|f\|_p^2 := \int_{T^m} |f|^p \, d\mu < \infty .$$

Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

$$\Rightarrow \quad \|fg\|_1 \leq \|f\|_2 \|g\|_2$$

In panticular, if $f \in L^2$ and $g \in L^2 \Rightarrow$

$$\Rightarrow f\bar{g} \in L^1$$

Define
$$\langle f, g \rangle = \int f\bar{g}\, d\mu.$$

This is an inner product on $L^2(\mu)$.

<u>Ex</u>: $L^2(\mathbb{T}^m)$ is a Hilbert space

<u>Proposition</u>: $\ell^2(\mathbb{Z}^m) \cong L^2(\mathbb{T}^m)$ as Hilbert spaces, i.e. $\exists$ lin. surjection $U : \mathcal{H} \to \mathcal{H}$ s.t. $\langle Uh, Ug \rangle = \langle h, g \rangle$ $\forall$ $h, g \in \mathcal{H}$.

Pf: (when $m=1$)

① A basis for $L^2(S^1)$ is given by

$$e_n(\theta) = e^{in\theta} \quad , \quad \theta \in [0, 2\pi]$$

or $\quad e_n(z) = z^n.$

For $n \in \mathbb{Z}$, define $\hat{f}(n) := \langle f, e_n \rangle$

$$= \int_{S^1} f(z) z^{-n} \, d\mu$$

then $\hat{f}(n)$ is the n^{th} Fourier coefficient of f

and

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n.$$

② Now $\hat{f} : \mathbb{Z} \to \mathbb{C}$ is an element of $\ell^2(\mathbb{Z})$.

by $\sum_{n=-\infty}^{\infty} \hat{f}(n) t^n$.

Define $\mathcal{U} : L^2(S^1) \longrightarrow \ell^2(\mathbb{Z})$

$$f \longmapsto \hat{f}$$

③ Parseval's inequality says that if $f = \sum a_n z^n$ is the Fourier series of f

$$\Rightarrow \sum |a_n|^2 = \int_{S^1} |f|^2 \, d\mu$$

Hence $\|Uf\| = \|f\|$.

(4) $Ue_n = \hat{e}_n = t^n$ so $\text{range}(U) \supseteq \mathbb{C}[\mathbb{Z}]$ so is dense in $\ell^2(\mathbb{Z})$. Since U is an isome. it is dense $\Rightarrow$ U surjective.

$\Rightarrow U$ is an isomorphism.

In fact $\ell^2(\mathbb{Z})$ and $L^2(S^1)$ are Hilbert spaces w/ same cardinality so they are isom..

$\blacksquare$

More generally

$$\ell^2(\mathbb{Z}^m) \cong L^2(\mathbb{T}^m)$$

Now there are many topologies on $B(\mathcal{H})$.

weak operator topology:

$$T_n \to T \iff \langle (T_n - T)x, y \rangle \to 0 \quad \forall \; x, y \in \mathcal{H}.$$

Strong operator topology:

$$T_n \to T \iff \|(T_n - T)x\| \to 0 \quad \forall \; x \in \mathcal{H}$$

norm topology

$$T_n \to T \iff \|T_n - T\| \to 0$$

Thm: ① $B(\mathcal{H})$ is complete with norm topology.

② $N(\Gamma)$ is the completion of $\mathbb{C}\Gamma$ is the weak operator topology.

Def: Let $L^\infty(T^m) =$ algebra of essentially bounded measurable functions on T^m (bounded modulo sets of measure 0)

We have an isomorphism

$$\begin{array}{ccc} L^\infty(T^m) & \xrightarrow{\cong} & N(\mathbb{Z}^m) = B(L^2(T^m))^{\mathbb{Z}^m} \\ g & \longmapsto & g \cdot f \end{array}$$

Here $\mathbb{Z}$ acts on T by

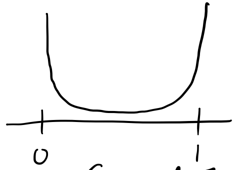
$$(z_1, \dots, z_m) \longmapsto (z_1^{a_1}, \dots, z_m^{a_m})$$

for $(a_1, \dots, a_m)$.

<u>Ex</u>: $tr(f) = \int_{S^1} f \, d\mu$

Compute $tr(a)$ $a \in \ell^2 \mathbb{Z}$.

Note: $L^2 S^1 \not\subseteq L^\infty S^1$ ex.



Compute trace of an element of $N\Gamma$.

Know that $f : S^1 \longrightarrow \mathbb{C}$ essentially bounded.

$$L^\infty(S^1) \longrightarrow N(L^2(S^1))$$

$$g \longmapsto L_g : f \longmapsto g \cdot f \qquad f \in L^2(S^1)$$

pointwise

can all be written as Four.

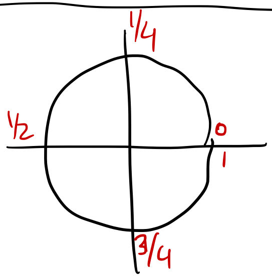
$$\mathrm{tr}(g) = \langle g(e), e \rangle = \langle g \cdot e, e \rangle$$

$$= \int_{S^1} g \, d\mu$$

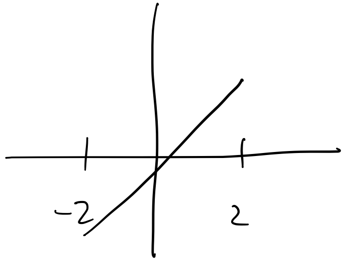
$$g = \sum_{n=-\infty}^{\infty} a_n z_n$$ where it is essentially bounded.

Ex where not in $\mathbb{C}\mathbb{Z}$?

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right)$$



$$\theta \longmapsto \theta$$



More generally,

$$\ell^2(\mathbb{Z}^m) \cong L^2(T^m) \quad \text{and}$$

$$N\mathbb{Z}^m \cong L^\infty(T^m)$$

and for $f \in L^\infty(T^m)$,

$$\operatorname{tr}_{N\mathbb{Z}^m}(f) = \int_{T^n} (f \cdot c_1) \cdot \bar{c}_1 \, d\mu$$

where $c_1(x) = 1$ is is the constant function

$$= \int_{T^n} f \, d\mu .$$