Dimension of arbitrary modules

Last time we defined $\dim_{N\Gamma}$ for any arbitrary $\dim_{N\Gamma}$ of a f.g. projective module by choosing a matrix $A \in M_n(N\Gamma)$ with $\operatorname{im}(A) =$ module and $A^2 = A$.

Want to show this is independent of choice.

This a standard trace argument. (see Lück).

<u>Note</u>: • $\dim_{N\Gamma}(P \oplus Q) = \dim_{N\Gamma}(P) + \dim_{N\Gamma}(Q)$

• $\dim_{N\Gamma}(P) = 0 \iff P = 0$

<u>Def</u>: Let M be an R-submodule of N. The <u>closure</u> of M is N is the R-submodule of N

$$\overline{M} = \{ x \in N \mid f(x) = 0 \ \forall f \in N^* \text{ with } M \subset \ker f \}$$

Here $N^* = \operatorname{Hom}_R(N, R)$ right R-module given by

$$(fr)(x) = f(x)r \quad \text{for } f \in N^*, \ x \in N, \ r \in R.$$

Define:

$$\mathbb{T}M := \{ x \in M \mid f(x) = 0 \ \forall f \in M^* \}$$

$$\mathbb{P}M := M / \mathbb{T}M$$

A sequence of R-mods $L \xrightarrow{i} M \xrightarrow{q} N$ is <u>weakly exact</u> if $\overline{\mathrm{im}(i)} = \ker(q)$.

- Note

$$TM = \overline{0}$$
$$TPM = 0$$
$$PPM = PM$$
$$M^* = (PM)^*$$
$$PM = 0 \iff M^* = 0.$$

Dimension for arbitrary $N\Gamma$-module, M.

<u>Theorem</u>: (1) If $K \subseteq M$ is a submodule of f.g. $N\Gamma$-mod. $M \Rightarrow M/\overline{K}$ is a f.g. projective module and $\overline{K}$ is a direct summand in M.

(2) If M is a f.g. $N\Gamma$-module, the PM is f.g. proj, $\exists$ an exact sequence

$$0 \to N\Gamma^n \longrightarrow N\Gamma^n \longrightarrow TM \to 0,$$

and

$$M \cong PM \oplus TM.$$

(3) $\exists$ a unique dim function

$$\dim_{N\Gamma} : \{N\Gamma\text{-modules}\} \longrightarrow [0, \infty]$$

s.t.

(a) If M is a f.g. projective module
$\Rightarrow$ it agrees with our prev. def.

(b) If $0 \to M_0 \to M_1 \to M_2 \to 0$ is exact.

$$\Rightarrow \quad \dim_{N\Gamma} M_1 = \dim_{N\Gamma} M_0 + \dim_{N\Gamma} M_2$$

(here $a + \infty = \infty$)

(c) If $M = \bigcup_{i \in I} M_i$ and for i, j $\exists k$ s.t.

$M_i, M_j \subseteq M_k$ (cofinal) $\Rightarrow$

$$\dim_{N\Gamma} M = \sup\{\dim_{N\Gamma} M_i \mid i \in I\}.$$

(d) If $\underset{\text{submod}}{K \subseteq M}$, M f.g. $\Rightarrow \dim_{N\Gamma} K = \dim_{N\Gamma} \overline{K}$.

<u>Remark</u>: Assuming this theorem, can see there is only one possibility for a module M. Let M be a $N\Gamma$-module and let $\{M_i\}$ be all f.g. submodules of M directed by inclusions.

Then by (c),

$$\dim_{N\Gamma}(M) = \sup\{\dim_{N\Gamma}(M_i) \mid i \in I\}.$$

and $\dim_{N\Gamma} M_i = \dim_{N\Gamma} PM_i$ since $PM_i = M_i$

Q. When is $\dim_{N\Gamma} M = 0$? Means that M has no non-trivial f.g. projective modules.

<u>Ex:</u> Γ = finite $\Rightarrow$ $N\Gamma = \mathbb{C}\Gamma$

$$tr_{N\Gamma}(\Sigma a_\gamma \gamma) = a_e$$

Show that for an $N\Gamma$-module V,

$$\dim_{N\Gamma}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}} V.$$

<u>Ex:</u> $\Gamma = \mathbb{Z}$ compute $\dim_{N\Gamma}(M)$, where M is a $N\mathbb{Z}$-module.