Def of L^2-Betti numbers

Last time we defined $\dim_{N\Gamma}$ for arbitrary modules. We saw this was similar to f.g. modules over $\mathbb{Z}$:

$$\mathbb{T}M \longleftrightarrow \text{torsion subgroup}(M)$$
$$\mathbb{P}M \longleftrightarrow \text{free summand}(M)$$
$$M = \mathbb{T}M \oplus \mathbb{P}M \longleftrightarrow M \cong \mathbb{Z}^r \oplus \mathbb{Z}/\langle p_1 \rangle \oplus \cdots \oplus \mathbb{Z}/\langle p_s \rangle$$

However, $\dim_{N\Gamma}$ is not necessarily an integer (ex. for Γ = finite).

* Moreover, can have non f.g. projective modules with finite $\dim_{N\Gamma}$!

Ex: Let $\Gamma = \mathbb{Z}^m$ then, $\ell^2\mathbb{Z}^m \cong L^2(T^m)$, $N(\mathbb{Z}^n) \cong L^\infty(T^n)$, and

$$tr(f) = \int_{T^n} f \, d\mu.$$

Let $\Omega \subseteq$ be a measurable set in T^n, and $\chi_\Omega : T^n \longrightarrow \mathbb{R}$ be the characteristic function, $\chi_\Omega(x) = \begin{cases} 0 & x \notin \Omega \\ 1 & x \in \Omega \end{cases}$.

Then the function M_Ω defined by

$$M_\Omega(g) = \chi_\Omega \cdot g \quad \leftarrow \text{mult. ptwise of functions}$$

is in $N(\mathbb{Z}^n)$ and

$$tr(M_\Omega) = \int_{T^n} \chi_\Omega \, d\mu = vol(\Omega).$$

Let $P_\Omega = im(N\mathbb{Z}^n \xrightarrow{(M_\Omega)} N\mathbb{Z}^m)$ be the image of the 1×1 matrix (M_Ω). Then P is projective since

$$\chi_\Omega \cdot \chi_\Omega(x) = \chi_\Omega(x),$$

and P_Ω is f.g.

Choose Ω_i with $vol(\Omega_i) = 1/2^i$, for $i \geq 0$, and let $P_i = P_{\Omega_i}$.

Then $P = \bigoplus_{i=0}^{\infty} P_i$ is projective but not finitely generated and

$$\dim_{N\mathbb{Z}^m}(P) = 1.$$

L^2-Betti numbers

Let X be a CW-complex and consider the chain complex of the universal cover, $\tilde{X}$

$$\to C_p(\tilde{X}) \xrightarrow{\tilde{\partial}} C_{p-1}(\tilde{X}) \to$$

Recall that $\pi_1(X)$ acts on the left on $\tilde{X}$, $\pi_1(X) \cong \mathrm{Deck}(\tilde{X} \to X)$. Hence $C_*(\tilde{X})$ is a left $\mathbb{Z}\Gamma$-module and $\tilde{\partial}_p$ are $\mathbb{Z}\Gamma$-module homomorphisms. Recall that if $r: S \to R$ is a ring homomorphism then R is an S-module via mult in R. (left or right).

$$s \cdot r := sr \in R,$$

for $s \in S$ and $r \in R$. In fact R is an R-S (or S-R) bimodule. Hence $N\Gamma$ is a $N\Gamma$-$\mathbb{Z}\Gamma$ bimodule, via $\mathbb{Z}\Gamma \to \mathbb{C}\Gamma \to N\Gamma$

Define $H_p(X; N\Gamma) := H_p(N\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{X}))$,

where $\partial_p = id \otimes \tilde{\partial}_p$. We note that $N\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{X})$ is a left $N\Gamma$-module and ∂_p is a left module hom, so $H_p(X; N\Gamma)$ is a left $N\Gamma$-module.

<u>Def</u>: The <u>p^{th} L^2-Betti number of X</u> by

$$b_p^{(2)}(X) := \dim_{N\Gamma}(H_p(X; N\Gamma)) \in [0, \infty].$$

Note if M is any right $\mathbb{Z}\Gamma$-module, one can define $H_p(X;M) := H_p(M \otimes_{\mathbb{Z}\Gamma} C_*(\hat{X}))$, this is called <u>homology with coefficients in M</u>.

One can also define L^2-Betti numbers for discrete groups. Let G be a discrete group and $K(G,1)$ be the classifying space. Define

$$b_p^{(2)}(G) := \dim_{NG} H_p(K(G,1); NG).$$

We recall that $N\Gamma$ is semi-hereditary. Thus if X is a finite CW-complex, then we can cellular chains to see that $H_p(X;N\Gamma)$ is a f.g. $N\Gamma$-module. It is not necessarily projective but $\exists$ ses

$$0 \longrightarrow P_i \longrightarrow Q_i \longrightarrow H_i(X;N\Gamma) \longrightarrow 0$$

for P_i, Q_i f.g. projective. It follows that

$$b_i^{(2)}(X) = \dim_{N\Gamma} Q_i - \dim_{N\Gamma} P_i.$$

<u>Properties</u>:

(1) If $X \underset{h.e.}{\simeq} Y \Rightarrow b_p^{(2)}(X) = b_p^{(2)}(Y)$.

(2) $\chi(X) = \sum (-1)^i b_i^{(2)}(X) = \sum (-1)^i b_i(p)$

(3) If $Y \xrightarrow{n:1} X$ is a finite cover of X then
$$b_p^{(2)}(Y) = n\, b_p^{(2)}(X).$$

(4) [P.D.] If X is a closed n-dim mfld, then
$$b_i^{(2)}(X) = b_{n-i}^{(2)}(X).$$

(5) $$b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(X)$$

(6) If X is connected then
$$b_0^{(2)}(X) = \begin{cases} 1/|\Gamma| & \text{if } |\Gamma| < \infty \\ 0 & \text{if } |\Gamma| = \infty \end{cases}$$

<u>Ex</u>

(1) $b_0^{(2)}(S^1) = b_1^{(2)}(S^1) = 0$

(2) Surfaces.

(3) Free groups