Whitehead torsion

Let R be an associative ring with unit. We have inclusions

$$\begin{aligned} GL_n(R) &\hookrightarrow GL_{n+1}(R) \hookrightarrow \cdots \\ M &\longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and $GL(R) = \varinjlim_n GL_n(R)$

<u>Def</u>: An matrix is elementary if it coincides with identity except for one off diagonal element.

<u>Lemma</u> (Whitehead): The subgp $E(R) \subseteq GL(R)$ gen by elementary matrices is $[GL(R), GL(R)]$. ← (commutator subgp)

<u>Pf</u> (easy): See Milnor's paper on "Whitehead torsion".

<u>Def</u>: The <u>Whitehead group</u> of R is

$$K_1(R) := GL(R)/E(R) = GL/[GL(R), GL(R)]$$

<u>Ex</u>: If $\mathbb{F}$ is a commutative field then

$$\det : K_1(\mathbb{F}) \xrightarrow{\cong} \mathbb{F}^* = \mathbb{F} \setminus \{0\}$$

is an isomorphism. This follows from fact that $SL(\mathbb{F})/E(\mathbb{F}) = 0$ since you can reduce every $\det = 1$ matrix to identity by elementary row operations.

<u>Ex</u>: Since $\mathbb{Z}$ has a Euclidean algorithm

$$K_1(\mathbb{Z}) = \{\pm 1\} \leftarrow \text{units in } \mathbb{Z}.$$

Ex: If F is a skew field (e.g. $\mathcal{K}(\Gamma)$ = right ring of fractions of $\mathbb{Z}\Gamma$, for Γ = PTFA group), then $K_1(F) = \mathcal{U}/[\mathcal{U},\mathcal{U}]$ where $\mathcal{U}$ = group of units of F.

Using the non-commutative Dieudonné determinant, we get a map

$$GL(F) \longrightarrow K_1(F) = \frac{\mathcal{U}}{[\mathcal{U},\mathcal{U}]}.$$

For $F = \mathcal{K}(\Gamma)$, with Γ = PTFA, $K_1(\mathcal{K}(\Gamma)) = \dfrac{\mathcal{K}^*}{[\mathcal{K}^*,\mathcal{K}^*]}$

where $\mathcal{K}^* = \mathcal{K}(\Gamma)^* = \mathcal{K}(\Gamma) \setminus \{0\}$.

Rmk: The map
$$\begin{aligned} GL(R) &\longrightarrow K_1(R) \\ A &\longmapsto [A] = \text{equiv class of } A \end{aligned}$$
can thus be thought of as a universal determinant for R.

Def: If G is a group, the Whitehead group of G

$$Wh(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Let C_* be a projective R-chain complex.

Lemma: A projective R-chain complex is acyclic $\Leftrightarrow$ it is contractible (i.e. has a chain contraction).

<u>Note</u>: contractible $\Rightarrow$ acyclic

but converse not true if not projective.

<u>Ex</u>: $R = \mathbb{Z}$ and consider

$$C_* = \underset{3}{0} \to \underset{2}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{1}{\mathbb{Z}} \xrightarrow{\pi} \underset{0}{\mathbb{Z}/2} \longrightarrow \underset{-1}{0}$$

Then $H_p(C_*) = 0 \ \forall\ p$. but has no contraction.

Suppose C_* is acyclic with contraction $\delta : C_* \to C_{*+1}$ s.t. $\partial \circ \delta + \delta \circ \partial = id$. Then

$$(\delta + \partial)|_{odd} : C_{odd} \longrightarrow C_{even}$$

is an isomorphism so after choosing a bases for $C_p\ \forall\ p$,

$$(\partial + \delta)_{odd} \in \tilde{K}_1(R).$$

where $\tilde{K}_1(R) = K_1(R)/\{(\pm 1)\}$. This does not depend on the choice of contraction. To see this, let γ be another chain contraction of C_* and let

$$\theta_i = \gamma_{i+1} \circ \delta_i : C_i \longrightarrow C_{i+2}.$$

Then $\theta_{even} : C_{even} \longrightarrow C_{even}$ and $\theta_{odd} : C_{odd} \longrightarrow C_{odd}$.

Note that $$\theta_i + id : C_i \longrightarrow C_{i+2}$$

off diagonal

so $$(\theta + id)_{odd} = \begin{array}{cc} \begin{pmatrix} I & \theta_0 & & \\ & I & \theta_2 & \\ & & & \ddots \end{pmatrix} & \begin{array}{l} \leftarrow C_0 \\ \leftarrow C_2 \\ \\ \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{basis for } C_0 & \text{basis for } C_2 \end{array} & \end{array}$$

Similarily for $\theta + id_{even}$. Hence $[\theta + id_{even}] = [\theta + id]_{odd}$
$= 0$ in $K_1(R)$

<u>Note</u>: $$\begin{tikzcd} C_{odd} \arrow[r, "\partial+\gamma"] \arrow[rrr, bend right, "id+\theta \;\cong"'] & C_{even} \arrow[r, "id+\theta", "\cong"'] & C_{even} \arrow[r, "\partial+\delta"] & C_{odd} \end{tikzcd}$$

Can also exchange order of γ and δ:

$$\begin{tikzcd} C_{odd} \arrow[r, "+\delta"] \arrow[rrr, bend right, "id+\theta' \;\cong"'] & C_{even} \arrow[r, "id+\theta'", "\cong"'] & C_{even} \arrow[r, "+\gamma"] & C_{odd} \end{tikzcd}$$

with $\theta' = \delta \circ \gamma$ $\Rightarrow (\partial + \gamma)_{even}$ and $(\partial + \delta)_{odd}$ are isomorphisms and

$$[\partial + \gamma_{even}] = -[\partial + \delta_{odd}] \in K_1(R).$$

Since $(\partial + \gamma)_{odd} (\partial + \gamma)_{even} = \partial\delta + \delta\partial = id$

$$\Rightarrow [\partial + \gamma_{odd}] = -[\partial + \gamma_{odd}]$$

$\blacksquare$

Def: For C_* a based acyclic free acyclic R-chain complex,

$$\tau(C_*) := [(\partial+\gamma)_{odd} : C_{odd} \to C_{even}] \in \widetilde{K}_1(R),$$

called the torsion of C_*.

Note: Need to work in $K_1(R)$ since bases are unordered.

Now suppose $\psi : C_* \to D_*$ be an R-chain map finite based free R-chain complexes. Then we can form the mapping cone:

$$\begin{array}{c} \mathrm{cone}(\psi) = \cdots \longrightarrow C_p \oplus D_{p+1} \xrightarrow{\partial = \begin{pmatrix} -\partial_C & 0 \\ \psi & d \end{pmatrix}} C_{p-1} \oplus D_p \longrightarrow \cdots \\ \shortparallel \\ \mathrm{cone}(\psi)_p \end{array}$$

i.e. $\partial(a,b) = (-\partial a, \partial b + \psi(a))$

Check $\partial^2(a,b) = (\partial^2 a, -\psi(\partial(a)) + \partial^2 b + \partial \circ \psi(a))$

$$= (0,0)$$

since ψ is a chain map.

Lemma (HW) If ψ is an R-chain homotopy equivalence, i.e. $\exists\ \phi$ s.t. $\psi\circ\phi$ and $\phi\circ\psi$ are chain homotopic to the identity, then cone(ψ) is acyclic.

$$\left[\begin{array}{l} \exists\ h \text{ s.t. } \partial h + h\partial = \psi\circ\phi - id \quad + \\ h' \text{ s.t. } \partial h' + h'\partial = \phi\circ\psi - id. \end{array}\right]$$

Def: The Whitehead torsion of ψ is

$$\tau(\psi) = \tau(\text{cone}(\psi)) \in \tilde{K}_1(R).$$

Whitehead torsion for homotopy equivalences:

Suppose $f: X \longrightarrow Y$ is a homotopy equivalence of finite connected CW-complexes. Then f lifts to

$$\tilde{f}: \tilde{X} \longrightarrow \tilde{Y}$$

of the universal covers, and we get

$$\tilde{f}_*: C_*(\tilde{X}) \longrightarrow C_*(\tilde{Y})$$

a $\mathbb{Z}G$-chain homotopy equivalence where $G = \pi_1(X) = \pi_1(Y)$

Choose a lift of each cell in X, Y we get bases for $C_*(\tilde{X})$ and $C_*(Y)$.

<u>Def</u>: Define the <u>Whitehead torsion</u> of the homotopy equivalence $f : X \to Y$ of finite connected CW complexes to be

$$\tau(f) := \tau(\tilde{f}_*) = [\mathrm{cone}(\tilde{f}_*)] \in \tilde{K}_1(G)$$

- If $f : X \to Y$ is a homeomorphism $\Rightarrow$

$$\tau(f) = 0.$$

In fact it vanishes precisely for "simple homotopy equivalences".