Homework Solutions 8 - Fall 2005 - Math 401

(1) p. 185 #3

Let $F: M \to N$ be a regular mapping and N be orientable. Then there is a 2-form η on N such that $\eta(q) \neq 0$ for all $q \in N$. Consider the 2-form $F^*\eta$ on M. Since $\eta(F(p)) \neq 0$, there are vectors $v, w \in T_{F(p)}N$ s.t. $\eta(v,w) \neq 0$. Moreover since F is regular, F_* is a 1-1 map between 2-dimensional vector spaces. Thus F_* is an isomorphism so there are vectors $\bar{v}, \bar{w} \in T_pM$ s.t. $F_*(\bar{v}) = v$ and $F_*(\bar{w}) = w$. Therefore

$$F^*\eta(\bar{v}, \bar{w}) = \eta(F_*(\bar{v}), F_*(\bar{w})) = \eta(v,w) \neq 0$$

so $F^*\eta$ is non-zero at p. Since p was arbitrary, F is a nowhere zero 2-form on M hence M is orientable.

(2) p. 185 #5 ; answer in back of book

(3) p. 185 #8 ; Counterexample to ...

(a) ... converse of (a) of Ex 2:

Let $N = \{(x,y,z) \mid z=0\}$, $M = \{(x,y,z) \mid z=0 \text{ or } z=2\}$, and

$F(x,y,z) = (x,y,0)$ then F is a surjective mapping, N = connected but M is not connected.

... converse of (b) of Ex 2:

Let $M = \{(u,v,0) \mid u,v \in \mathbb{R}\} = \mathbb{R}^2$; N = torus = $\{x(u,v) \mid u,v \in \mathbb{R}\}$ where

$x(u,v) = ((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u)$ and

$F(u,v,0) = x(u,v)$. Then F is a surjective mapping and N is compact but M is not compact.

(b) ... Ex 3 when F is not regular.:

Let M be any non-orientable surface (like a Mobius band), N be any orientable surface (like S^2) and $F(p) = q_0$ be a constant map. Then F is a mapping and N is orientable but M is non-orientable.

(c) ... converse of Ex 3:

Let N be any non-orientable surface (like the Mobius band) and $x : D \longrightarrow N$ be a patch where $D \subset \mathbb{R}^2$. Then D is orientable and x is regular so let $M = D$ and $F = x$.

(4) p. 200 #1

Let α be a curve in M and U be a unit normal of M restricted to α. Then

$$S(\alpha') = -\nabla_{\alpha'} U = -(U \circ \alpha)' = -U' \qquad \text{by method 1}$$

(5) p. 200 #4

Image of Gauss map:

(a) Cylinder

Image is a great circle

(b) Cone

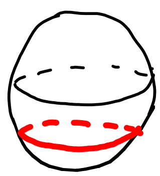

Image is a circle

(c) $\mathbb{R}^2$

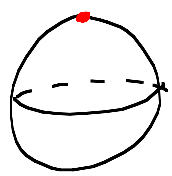

Image is a point

(d) sphere

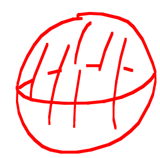

Image is entire sphere

(6) p. 201 #9

Let V be a tangent vector field on M (w/ unit normal U). Suppose W is another tangent vector field on M. Then

$$\nabla_V W \cdot U + W \cdot \nabla_V U = V[W \cdot U] = V[0] = 0$$

since W is orthogonal to U. Hence

$$S(V) \cdot W = -\nabla_V U \cdot W = \nabla_V W \cdot U.$$

Define $[V, W] = \nabla_V W - \nabla_W V$ for two tangent vector fields V, W on M.

We calculate that

$$\begin{aligned} [V, W] \cdot U &= (\nabla_V W - \nabla_W V) \cdot U \\ &= \nabla_V W \cdot U - \nabla_W V \cdot U \\ &= S(V) \cdot W - S(W) \cdot V \quad \text{from above.} \end{aligned}$$

Thus S is symmetric if and only if $[V, W]$ is a tangent v.f. on M.
$([V, W] \cdot U = 0)$