Homework 5 Solutions - Fall 2005 - Math 401

(1) p. 81 #2(d,f)

(d) $\nabla_W V = \cos x \nabla_{U_1} V + \sin x \nabla_{U_2} V$, $\nabla_{U_1} V = U_1[-y]U_1 + U_1[x]U_3 = U_3$,

$\nabla_{U_2} V = U_2[-y]U_1 + U_2[x]U_3 = -U_1$

$\Rightarrow \nabla_W V = \cos x\, U_3 - \sin x\, U_1$

(f) $\nabla_V(xV - zW) = -y\nabla_{U_1}(xV - zW) + x\nabla_{U_3}(xV - zW)$,

$xV - zW = -(xy + z\cos x)U_1 - z\sin x\, U_2 + x^2 U_3$

$$\Rightarrow \nabla_{U_1}(xV - zW) = -U_1[xy + z\cos x]U_1 - U_1[z\sin x]U_2 + U_1[x^2]U_3$$
$$= -(y - z\sin x)U_1 - z\cos x\, U_2 + 2x\, U_3$$

$$\nabla_{U_3}(xV - zW) = -U_3[xy + z\cos x]U_1 - U_3[z\sin x]U_2 + U_3[x^2]U_3$$
$$= -\cos x\, U_1 - \sin x\, U_2$$

$$\Rightarrow \nabla_V(xV - zW) = [y(y - z\sin x) - x\cos x]U_1 + (yz\cos x - x\sin x)U_2 - 2xy\, U_3$$

(2) p. 81 #3; Let W be a v.f. with constant length $\|W\|$.

Then $\|W\|^2 = W \cdot W = c$ constant so $v[W \cdot W] = v[\text{constant}] = 0$ for all vectors v. Let V be a vector field.

Hence by Thm 5.3 (4), for all p,

$$0 = V(p)[W \cdot W] = \nabla_{V(p)} W \cdot W(p) + W(p) \cdot \nabla_{V(p)} W = 2\nabla_{V(p)} W \cdot W(p)$$

Thus $(\nabla_V W \cdot W)(p) = \nabla_{V(p)} W \cdot W(p) = 0$ for all p hence

$\nabla_V W \cdot W = 0$ so $\nabla_V W$ is everywhere orthogonal to W.

(3) p. 81 #5; Let W be a vector field on a region containg a regular curve α. Define $W_\alpha(t) = W(\alpha(t))$, then

$W_\alpha(t) = W(\alpha(t)) = \sum w_i(\alpha(t))U_i$ where $W = \sum w_i U_i$.

(a) $\nabla_{\alpha'(t)} W = \sum \alpha'(t)[W_i] U_i = \sum \frac{d(w_i(\alpha(t)))}{dt}(t)\, U_i$ by Lemma 4.6.

$$= \sum_i w_i(\alpha(t))'\, U_i$$

$$= W_\alpha'(t)$$

(b) Let W be a vector field on $\mathbb{R}^3$ and let v_p be a tangent vector to $\mathbb{R}^3$ at the point p. Define

$$\tilde{\nabla}_{v_p} W = W(\alpha(t))'(0)$$

if α is any curve in $\mathbb{R}^3$ with $\alpha(0) = p$ and $\alpha'(0) = v_p$.

Since $(W_\alpha)'(0) = W(\alpha(t))'(0) = \nabla_{\alpha'(0)} W = \nabla_{v_p} W$ by part (a),

it follow that if α and β are any two curves with $\alpha(0) = p = \beta(0)$ and $\alpha'(0) = v_p = \beta(0)$ then

$$W(\alpha(t))'(0) = \nabla_{v_p} W = W(\beta(t))'(0)$$

so $\tilde{\nabla}_{v_p} W$ is well-defined.

Moreover $\tilde{\nabla}_{v_p} W = W_\alpha'(0) = \nabla_{v_p} W$, so $\tilde{\nabla}$ is the same as ∇. Hence we can replance the straight line $p+tv$ in definition 5.1 with any curve α s.t. $\alpha(0)=p$ and $\alpha'(0) = v_p$.

(4) p.89 #2; Find the connection forms for U_1, U_2, U_3.

$$a_{ij} = U_i \cdot U_j = \delta_{ij} \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow dA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \omega_{ij} = 0 \text{ for all } i,j.$$

(5) p.90 #5; Let $F_1 = \cos\varphi(\cos\theta\, U_1 + \sin\theta\, U_2) + \sin\varphi\, U_3$
$F_2 = -\sin\theta\, U_1 + \cos\theta\, U_2$ and
$F_3 = -\sin\varphi(\cos\theta\, U_1 + \sin\theta\, U_2) + \cos\varphi\, U_3$ be the

spherical frame field. Then

$$A = (F_i \cdot U_j) = \begin{pmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta & \sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{pmatrix}$$

$$\Rightarrow\ dA = \begin{pmatrix} -\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta & -\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta & \cos\varphi\, d\varphi \\ -\cos\theta\, d\theta & -\sin\theta\, d\theta & 0 \\ -\cos\varphi\cos\theta\, d\varphi + \sin\varphi\sin\theta\, d\theta & -\cos\varphi\sin\theta\, d\varphi - \sin\varphi\cos\theta\, d\theta & -\sin\varphi\, d\varphi \end{pmatrix}$$

$$w_{12} = (dA\cdot A^T)_{12} = (\cancel{-\sin\varphi\cos\theta\, d\varphi} - \cos\varphi\sin\theta\, d\theta)(-\sin\theta) + (\cancel{-\sin\varphi\sin\theta\, d\varphi} + \cos\varphi\cos\theta\, d\theta)(\cos\theta)$$

$$= (\cos\varphi\sin\theta^2 + \cos\varphi\cos\theta^2)\, d\theta = \cos\varphi\, d\theta \quad ✓$$

$$w_{13} = (-\sin\varphi\cos\theta\, d\varphi - \cos\varphi\sin\theta\, d\theta)(-\sin\varphi\cos\theta) + (-\sin\varphi\sin\theta\, d\varphi + \cos\varphi\cos\theta\, d\theta)(-\sin\varphi\sin\theta)$$
$$+ \cos\varphi\, d\varphi\cdot\cos\varphi$$

$$= (\sin\varphi^2\cos\theta^2 + \sin\varphi^2\sin\theta^2 + \cos\varphi^2)\, d\varphi + \underbrace{(\cos\varphi\sin\varphi\cos\theta\sin\theta - \cos\theta\sin\theta\cos\varphi\sin\varphi)}_{=0} d\theta$$

$$= (\sin\varphi^2 + \cos\varphi^2)\, d\varphi = d\varphi \quad ✓$$

$$w_{23} = -\cos\theta\, d\theta(-\sin\varphi\cos\theta) - \sin\theta\, d\theta(-\sin\varphi\sin\theta) + 0$$
$$= (\sin\varphi\cos\theta^2 + \sin\varphi\sin\theta^2)\, d\theta$$
$$= \sin\varphi\, d\theta \quad ✓$$

(6) p.90 #8 ; Let β be a unit speed curve in $\mathbb{R}^3$ with $\kappa > 0$ and suppose E_1, E_2, E_3 is a frame field s.t. $E_1(\beta(t)) = T(t)$, $E_2(\beta(t)) = N(t)$, $E_3(\beta(t)) = B(t)$.

Then $\nabla_{T(t)} E_1 = \nabla_{\beta'(t)} E_1 = E_1(\beta(t))'(t)$ by problem 5(a)

But $E_1(\beta(t))'(t) = T'(t) = \kappa(t) N(t) = \kappa(t)\cdot E_2(\beta(t))$

by the frenet equations.

But $\omega_{12}(T(t))$ is the 1-form s.t.

$$\nabla_{T(t)} E_1 = \omega_{12}(T(t))\, E_2(\beta(t)) \quad \text{by theorem 7.2}$$

so $\omega_{12}(T(t)) = \kappa(t)$ hence $\omega_{12}(T) = \kappa$.

Also $\omega_{13}(T(t))$ is the 1-form in front of $E_3(\beta(t))$ in the expansion of $\nabla_{T(t)} E_1$ so $\omega_{13}(T) = 0$.

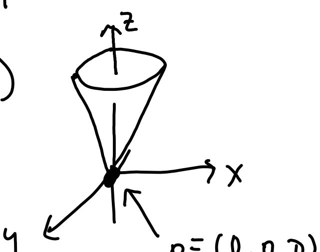
We use a similar arguments for $\omega_{23}(T)$.

$$\nabla_{T(t)} E_2 = E_2(\beta(t))'(t) = N'(t) = -\kappa(t)T(t) + \tau B(t) = -\kappa(t) E_1(\beta(t)) + \underline{\underline{\tau(t)}}\, E_3(\beta(t))$$

so $\omega_{23}(T) = \tau$.

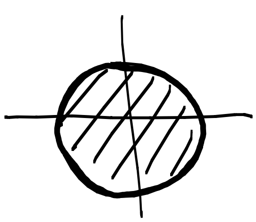
(7) p. 132 #1

(a)



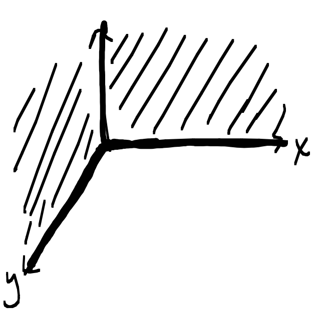
It is impossible to find a proper patch about $p = (0,0,0)$

(b)



cannot find a patch around any of the points $\{(x,y,0) \mid x^2+y^2 = 1\}$

(c)



no patches around points in

$\{(x,0,0), x \geq 0\} \cup \{(0,y,0) \mid y \geq 0\} \cup \{(0,0,z) \mid z \geq 0\}$

(8) p.132 #5(a,b)

(a) $M = \{g = 0\}$ where $g(x,y,z) = (x^2+y^2)^2 + 3z^2 - 1$

$$dg = 2(x^2+y^2)[2x\,dx + 2y\,dy] + 6z\,dz$$

Let (x_0, y_0, z_0) be a point of M then $(x_0^2+y_0^2)^2+3z_0^2=1$.

If $z_0=0$ then $x_0^2+y_0^2=1$ so both x_0 and y_0 are not both zero hence

$$dg(x_0,y_0,0) = 2x_0\,dx + 2y_0\,dy \neq 0.$$

If $z_0 \neq 0$ then $dg(x_0,y_0,z_0) = \underline{\quad}\,dx + \underline{\quad}\,dy + 6z_0\,dz \neq 0$.

Hence dg is never 0 for any point on M so by Theorem 1.4, M is a surface.

(b) $M_c = \{g=c\}$ such that $g(x,y,z) = z(z-2)+xy$

$dg = y\,dx + x\,dy + 2(z-1)\,dz$ so

$dg(x_0,y_0,z_0) = 0$ if and only if $y_0=0$, $x_0=0$, $z_0=1$.

Thus M_c is a surface provided $(0,0,1) \notin M_c$.

$g(0,0,1) = 1(1-2)+0 = -1$ so if $c \neq -1$ then M_c is a surface.

When $c=-1$, the M_c looks like (not a surface point).