MATH 401 - FALL 2005 - MIDTERM EXAM
DUE IN CLASS ON WEDNESDAY, OCTOBER 19

This is a pledged, timed exam and must be completed in 2 continuous hours. You may use the book "Elementary Differential Geometry" by B. O'Neill, and your personal class notes, but of course, you must work alone. You may use Maple, Matlab or Mathematica for computations, but please print out your work and include it with your exam if you do so. You may not use the internet or any other books, notes or references. Please show all of your work.

I will post corrections or clarifications (hopefully there will not be any to post) on the course website. At any point in the exam, you may check the course website to see if there are any corrections for the exam. If you think there is a mistake on the exam and it is not corrected on the website, you may call me at x3659 or 713-443-6113 (only between the hours of 9am-11pm).

Please remember to write your name on the exam, sign the pledge, and record your starting and ending times below.

Name:

Pledge:

Starting Time:

Ending Time:

Good Luck!!!

1. Let $\alpha(t) = (2 + 5\sin(t), 3 + 4\cos(t), 1 + 3\cos(t))$ for $t \in \mathbb{R}$.
   (a) Compute the speed $v(t)$ of α.
   (b) Compute the Frenet apparatus of α.
   (c) Describe the shape of the image of α. How do you know this?
   (d) Find a unit-speed reparametrization of α.

2. Let α be a unit speed curve in R^3. Find a formula for α''' in terms of κ (curvature of α), κ', τ (torsion of α), N (normal to α), T (tangent to α), and B (binormal of α).

3. Let $f(x, y, z) = x^2 + yz$, $\phi = yz\, dx + dz$ and $\xi = \sin(z)dx + \cos(z)dy$. Find the standard expression (in terms of $dx\, dy$, etc.) for
   (a) $\phi \wedge \xi$
   (b) $d\phi$
   (c) $d(f\phi)$.

4. Let $W = \sum_{i=1}^{3} x_i U_i$ where x_1, x_2 and x_3 are the natural coordinate functions on $\mathbb{R}^3$. Show that $\nabla_V W = V$ for every (differentiable) vector field V on $\mathbb{R}^3$.

5. Given a 1-form ϕ on $\mathbb{R}^3$, we can define a 2-form $*\phi$ by setting

$$*dx = dy\, dz$$
$$*dy = dz\, dx$$
$$*dz = dx\, dy$$

and extending by linearity (i.e. $*(f_1\phi_1 + f_2\phi_2) = f_1(*\phi_1) + f_2(*\phi_2)$ for any functions f_i and 1-forms ϕ_i). $*$ is called the *Hodge star operator.*

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The Laplacian of f, denoted Δf, is defined by $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. Show that $d*(df) = (\Delta f)\, dx\, dy\, dz$.

6. Prove that the definition of $\mathbf{v}[f]$ in Definition 3.10 (p. 150 of O'Neill) is well defined. More specifically, let M be a surface in $\mathbb{R}^3$, $\mathbf{v}$ be a tangent vector to M at $\mathbf{p}$ and $f : M \to \mathbb{R}$ be a differentiable function on M. Suppose $\alpha : I \to M$ and $\beta : I \to M$ are two curves in M such that $\alpha(0) = \mathbf{p} = \beta(0)$ and $\alpha'(0) = \mathbf{v} = \beta'(0)$. Show that $\frac{d}{dt} f(\alpha(t))_{|t=0} = \frac{d}{dt} f(\beta(t))_{|t=0}$.
Hint: Use Lemma 3.1 on p. 145 of O'Neill.