Solutions Homework #1 Math 401 - Fall 2005

(1) p. 15 #3(f): $f = xy$, $V = y^2 U_1 - x U_3$ so by Lemma 3.2,

$V[f] = \sum v_i \frac{\partial f}{\partial x_i} = y^2 y + 0$ and hence

$\boxed{V[V[f]] = 0}$

(2) p. 15 #5: Suppose $V[f] = W[f]$ for all f on $\mathbb{R}^3$. Then in particular, $V[x_i] = W[x_i]$ for all coordinate functions $x_i \mid 1 \le i \le 3$. Therefore by problem #4,

$$V = \sum V[x_i] U_i = \sum W[x_i] U_i = W.$$

(3) p. 39 #5: By definition, $F_*(v_p) = \beta'(0)$ where $\beta(t) = F(p + tv)$. Since F is linear, $\beta(t) = F(p) + tF(v)$. Thus, $F_*(v_p) = \beta'(0) = \frac{d}{dt}(F(p) + tF(v)) = F(v)_{F(p)}$.

(4) p. 40 #8: Let $v_p \in T_p\mathbb{R}^3$. Define $\bar{F}_*(v_p)$ by the following: pick <u>any</u> curve $\bar{\alpha}: I \longrightarrow \mathbb{R}^3$ s.t. $\bar{\alpha}'(0) = v_p$. Let $\bar{F}_*(v_p) = \bar{\beta}'(0)$ where $\bar{\beta} = F \circ \bar{\alpha}$. We will show that this is a well-defined function and is equal to F_* for all v_p. To accomplish this, we will show that $\bar{F}_*(v_p) = F_*(v_p)$.

Let $\bar{\alpha}$ be any curve with $\bar{\alpha}'(0) = V_p$. By Corollary 7.7, $\bar{\beta}'(0) = F_*(\bar{\alpha}'(0))$. Therefore,

$$\begin{aligned} \bar{F}_*(V_p) &\overset{def}{=} \bar{\beta}'(0) \\ &= F_*(\bar{\alpha}'(0)) \\ &= F_*(V_p) . \end{aligned}$$

(5) p.40 #9, By assumption, all derivatives of f_i and g_i exist and are continuous, where $F = (f_1, \dots, f_m)$ and $G = (g_1, \dots, g_p)$.

Write $F \circ G = (f_1(g_1, \dots, g_p), f_2(g_1, \dots, g_p), \dots, f_m(g_1, \dots, g_p))$.

(a) By calculus, we know that all first partials of $h_i := f_i(g_1, \dots, g_p)$ exist and are continuous.

Moreover

$\frac{\partial h_i}{\partial x_j} = \frac{\partial f_i(g_1, \dots, g_p)}{\partial x_j}$ is a sum and product of terms $\frac{\partial f_i}{\partial u_k}(g_1, \dots, g_p)$ and $\frac{\partial g_k}{\partial x_j}$ hence the partials of $\frac{\partial h_i}{\partial x_j}$ exist and are continuous.

But the first partials of $\frac{\partial h_i}{\partial x_j}$ are precisely the second partials of $f_i(g_1, \dots, g_p)$. Continue in this way we prove that the partials of $f_i(g_1, \dots, g_p)$ of all orders exist and are continuous, Hence $F \circ G$ is differentiable.

(b) Let $V_p \in T_p\mathbb{R}^3$, then let $\alpha: I \longrightarrow \mathbb{R}^3$ s.t.,

$\alpha'(0) = V_p$. By the previous exercise,

$$(F\circ G)_*(V_p) = \beta'(0) \qquad \text{where } \beta = (F\circ G)\circ\alpha \text{ and}$$

$$G_*(V_p) = \gamma'(0) \qquad \text{where } \gamma = G\circ\alpha.$$

Since γ is a curve s.t. $\gamma'(0) = G_*(V_p)$,

$$F_*(G_*(V_p)) = \delta'(0) \qquad \text{where } \delta = F\circ\gamma = F\circ(G\circ\alpha).$$

However, since composition is associative,

$$\beta = F\circ(G\circ\alpha) = (F\circ G)\circ\alpha = \delta \quad \text{hence}$$

$$\beta'(0) = \delta'(0) \Rightarrow (F\circ G)_*(V_p) = F_*(G_*(V_p)).$$

(c) This is immediate from the definition.

Also note that by part (b)

$$id = id_* = (F\circ F^{-1})_* = F_*\circ(F^{-1})_* \Rightarrow (F^{-1})_* = (F_*)^{-1}.$$

16) p.26 #1:

(a) $(y^2\,dx)(V_p) = y^2(p)\cdot v_1 = (-2)^2\cdot 1 = \boxed{4}$

(b) $(z\,dy - y\,dz)(V_p) = z(p)\cdot V_2 - y(p)\cdot V_3 = 1\cdot 2 - (-2)(-3) = \boxed{-4}$

(c) $((z^2-1)\,dx - dy + x^2\,dz)(V_p) = (1-1)1 - 2 + 0^2(-3) = \boxed{-2}$