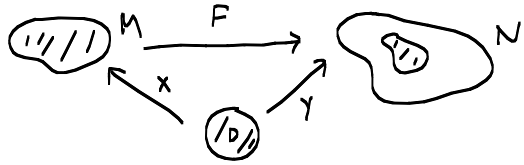
Homework 7 - Solutions - Fall 2005 - Math 401

(1) p.167 #3

Suppose M is a simple surface defined by $x: D \longrightarrow \mathbb{R}^3$. Let y be a mapping into a surface N.



Define $F: M \longrightarrow N$ by $F = y \circ x^{-1}$ and let x_1 and y_1 be any two patches in M and N respectively with $\operatorname{im} x \cap \operatorname{im} x_1 \neq \emptyset$ (similar for y_1 and y). By Corollary 3.3, $x^{-1}x_1$ and $y_1^{-1}y$ are differentiable mappings. Therefore

$$(x^{-1}x_1)^{-1}y_1^{-1}y = x_1^{-1}Fy \quad \text{is differentiable so}$$

F is a differentiable function.

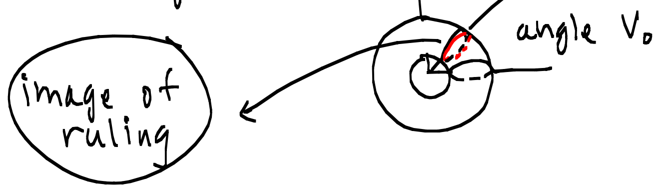
(2) Let H be the Helicoid given by $x(u,v) = (u\cos v, u\sin v, bv)$ where $b \neq 0$ and T be torus given by the parametrization

$y(u,v) = (h(u)\cos v, h(u)\sin v, g(u))$ where $h(u) = R + r\cos u$ and $g(u) = r\sin(u)$.

Define $F: H \longrightarrow T$ by $F = y \circ x^{-1}$. By the above exercise, if we restrict F to a small open set D in $\mathbb{R}^2$ s.t. $y|_D$ is a patch then $F|_D$ is differentiable. Since we can all of H by $x|_D$ where D is such an open, F is differentiable. Now, a ruling is when we fix a $v = v_0$ so

$$\begin{aligned} F(u,v_0) &= y \circ x^{-1}(u\cos v_0, u\sin v_0, bv) \\ &= y(u, v_0) \\ &= (h(u)\cos v_0, h(u)\sin v_0, g(u)) \end{aligned}$$

is a meridian of T (fixes an angle of the revolution)



(3) p.168 #12

(a) $PP^{-1}(u,v) = P\left[\dfrac{(4u, 4v, 2f)}{f+4}\right]$ where $f = u^2 + v^2$

$$= \left(\left(\frac{8u}{f+4}\right)\Big/\left(2 - \frac{2f}{f+4}\right), \left(\frac{8v}{f+4}\right)\Big/\left(2 - \frac{2f}{f+4}\right)\right)$$

$$= \left(\left(\frac{8u}{f+4}\right) \Big/ \left(\frac{\cancel{2f}+8-\cancel{2f}}{f+4}\right), \left(\frac{8v}{f+4}\right) \Big/ \left(\frac{\cancel{2f}+8-\cancel{2f}}{f+4}\right) \right)$$

$$= (u, v)$$

Let $(x,y,z) \in S^2$ (with center $(0,0,1)$ of radius 1) then $x^2+y^2+(z-1)^2=1$ hence

$$P^{-1}P(x,y,z) = P^{-1}\left(\frac{2x}{2-z}, \frac{2y}{2-z}\right)$$

$$= \frac{1}{f+4}\left(\frac{8x}{2-z}, \frac{8y}{2-z}, 2f\right) \quad \text{where } f = \left(\frac{2x}{2-z}\right)^2 + \left(\frac{2y}{2-z}\right)^2$$

$$= \frac{8}{(f+4)(2-z)}\left(x, y, \frac{f(2-z)}{4}\right)$$

Now $f = 4x^2+4y^2/(2-z)^2$ so $\frac{1}{4}f(2-z) = (x^2+y^2)/(2-z)$.

But $f+4 = (4x^2+4y^2+4(2-z)^2)/(2-z)^2 = 4[x^2+y^2+z^2-4z+4]/(2-z)^2$

$$= 4[-2z+4]/(2-z)^2 = 8/(2-z)$$

since $x^2+y^2+z^2-2z=0$ (x,y,z on S^2)

Thus $P^{-1}P(x,y,z) = \left(x, y, \frac{x^2+y^2}{2-z}\right)$.

We now show $\frac{x^2+y^2}{2-z} = z$ hence $P^{-1}P = id$:

$$\frac{x^2+y^2}{2-z} = \frac{x^2+y^2+z^2-2z+2z-z^2}{2-z} = \frac{z(2-z)}{2-z} = z.$$

(4) p.175 #3

(a) Let ϕ be closed and x be a 2-segment then by Stokes' Theorem

$$\int_{\partial x} \phi = \iint_x d\phi = \iint_x 0 = 0$$

(b) Let ϕ be exact and α be a piecewise smooth curve with smooth segments $\alpha_1, \dots, \alpha_k$. then $\phi = df$ so by Theorem 6.2,

$$\int_{\alpha_i} \phi = \int_{\alpha_i} df = f(\alpha_i(1)) - f(\alpha_i(0)).$$

Hence
$$\int_\alpha \phi = \sum_i \int_{\alpha_i} \phi = \sum_i f(\alpha_i(0)) - f(\alpha_i(0)) = 0$$

since $\alpha_i(1) = \alpha_{i+1}(0)$ for $1 \le i \le k-1$ and

$\alpha_k(1) = \alpha_1(0)$.

(5) p. 175 # 4

(a) Let $\alpha(t) = (\cos t, \sin t)$ on $0 \le t \le 2\pi$. Then $\alpha'(t) = -\sin t\, U_1 + \cos t\, U_2$ so

$dv(\alpha'(t)) = \cos t$ and $du = -\sin t$. Therefore

$$\psi(\alpha'(t)) = \frac{\cos t \cdot \cos t}{\cos t^2 + \sin t^2} - \frac{\sin t\,(-\sin t)}{\cos t^2 + \sin t^2} = 1$$ and

$$\int_\alpha \psi = \int_0^{2\pi} \psi(\alpha'(t))\, dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

Hence ψ is not exact.

However ψ is closed since

$$d\psi = d\left(\frac{u}{u^2+v^2}\right) \wedge dv - d\left(\frac{v}{u^2+v^2}\right) \wedge du$$

$$= \frac{\partial f}{\partial u}\, du\, dv - \frac{\partial g}{\partial v}\, dv\, du \qquad \text{where } f = \frac{u}{u^2+v^2} \text{ and } g = \frac{v}{u^2+v^2}.$$

$$= \frac{v^2-u^2}{(u^2+v^2)^2}\, du\, dv + \frac{(u^2-v^2)}{(u^2+v^2)}\, du\, dv$$

$$= 0$$

(b) We wish to show that ψ is exact on $\underbrace{\{(u,v) \mid v > 0\}}^{=\mathcal{U}}$. To do this, we find an $f : \mathcal{U} \to \mathbb{R}$ s.t. $df = \psi$. Let

$f = -\arctan(u/v)$. This is defined as long as $v \neq 0$,

so for instance, f is defined on $\mathcal{U}$. Then

$$df = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv = \frac{-v}{u^2+v^2}\, du + \frac{u}{u^2+v^2}\, dv = \psi$$

(6) p. 176 # 7

(a) see back of book

(b) Should read: ... and η is a 2-form on N, then $\int_X F^*\eta = \int_{F(X)} \eta$.

Let X be a 2-segment in M and η be a 2-form on N then

$$\int_X F^*\eta = \iint_R F^*\eta\,(X_u, X_v)\,du\,dv$$

$$= \iint_R \eta\big(F_*(X_u), F_*(X_v)\big)\,du\,dv$$

$$= \iint_R \eta\big((F\circ X)_u, (F\circ X)_v\big)\,du\,dv$$

$$= \iint_{F\circ X} \eta$$