Solutions - Homework 3 - M 401 Fall 2005

(1) p.49, # 9

(a) Let $S = \{p \in \mathbb{R}^3 \mid \|p\| < 1\}$ then S is an ε-neighborhood of $\vec{0}$ in $\mathbb{R}^3$ with $\varepsilon = 1$. Therefore every point in S is contained in an ε-nbhd of 0 contained in S, and hence S is open

(b) Let $S = \{p \in \mathbb{R}^3 \mid p_3 > 0\}$ and $N_{p_3}(p) = \{q \in \mathbb{R}^3 \mid d(p,q) < p_3\}$ then $N_{p_3}(p)$ is an ε-nbhd of p for all $p \in S$, and $p \in N_{p_3}(p)$. We will show that $N_{p_3}(p) \subset S$ when $p \in S$ hence S is open. Let $q \in N_{p_3}(p)$ and $p \in S$ then

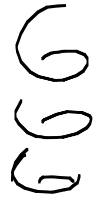
$$|p_3 - q_3| \leq d(p,q) < p_3$$

hence $q_3 > 0$. Therefore $q \in S$ so $N_{p_3}(p) \subset S$.

(2) p. 55 # 2

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a curve. Then $v(t)$ is constant if and only if $\|\alpha(t)\|^2$ is constant. But $\|\alpha(t)\|^2$ is constant if and only if $d/dt\, \|\alpha(t)\|^2 = 0$. Since $\frac{d}{dt} \|\alpha(t)\|^2 = 2\alpha'(t) \cdot \alpha(t)$, the speed of $v(t)$ is constant if and only if $\alpha'(t) \cdot \alpha(t) = 0$, or the velocity is orthogonal to $\alpha(t)$.

(3) p.55 #6



$\alpha(t) = (\cos t, \sin t, t)$

(a) $Y(t) = -\cos t\, U_1 - \sin t\, U_2 - t\, U_3$

(b) $Y(t) = \alpha'(t) - \alpha''(t)$

$= (-\sin t + \cos t)U_1 + (\cos t + \sin t)U_2 + U_3$

(c) $Y(t) = \dfrac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|}$

$= \dfrac{\sin t\, U_1 - \cos t\, U_2 + 1}{\sqrt{2}}$

(d) $Y(t) = (\cos(t+\pi) - \cos(t))U_1 + (\sin(t+\pi) - \sin(t))U_2 + \pi U_3$

(4) p.56 #10

Suppose $\alpha'(t)$ and $\beta'(t)$ are parallel for all $t \in I$.

Let $\alpha(t) = \sum \alpha_i(t) U_i(\alpha(t))$ and $\beta(t) = \sum \beta_i(t) U_i(\beta(t))$

then $\alpha_i'(t) = \beta_i'(t)$ for all $t \in I$ and $i = 1,2,3$.

Therefore $\alpha_i(t) = \beta_i(t) + c_i$ where $c_i \in \mathbb{R}$ hence

$\beta(t) = \alpha(t) + p$ where $p = (c_1, c_2, c_3)$ for all $t \in I$.

(5) p.56 #11

(a) Let $\sigma(t) = (1-t)p + tq$ for $0 \le t \le 1$. Then

$\|\sigma(t)\| = \sqrt{(-p_1+q_1)^2 + (-p_2+q_2)^2 + (-p_3+q_3)^2}$

$= \|p-q\|$

and hence

$$L(\sigma) = \int_0^1 \|\sigma'(t)\| dt = \|p-q\| = d(p,q)$$

(b) Since $\alpha' \cdot u = \|\alpha'\| \|u\| \cos\theta \leq \|\alpha'\|$ where θ is the angle between α' and u,

$$L(\alpha) = \int_0^1 \|\alpha'(t)\| dt \geq \int_0^1 \alpha'(t) \cdot u \, dt .$$

We write $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $u = (u_1, u_2, u_3)$ then

$$\begin{aligned} \int_0^1 \alpha'(t) \cdot u \, dt &= \sum u_i \int_0^1 \alpha_i'(t) \, dt \\ &= \sum (\alpha_i(1) - \alpha_i(0)) u_i \\ &= (q-p) \cdot u \\ &= (q-p) \frac{(q-p)}{\|q-p\|} \\ &= d(p,q) \end{aligned}$$

Hence $L(\alpha) \geq d(p,q)$.

(c) Suppose $L(\alpha) = d(p,q)$, then

$$\int_0^1 (\|\alpha'(t)\| - \alpha'(t) \cdot u) \, dt = 0$$

but $\|\alpha'(t)\| \geq \alpha'(t) \cdot u$ so $\|\alpha'(t)\| - \alpha'(t) \cdot u$ is a positive function with integrue 0. Therefore,

$\|\alpha'(t)\| = \alpha'(t)\cdot u$ for all $t \in I$.

We can write $\alpha'(t) = (\alpha'(t)\cdot u)\,u + Y(t)$ for some Y with $Y(t)\cdot u = 0$. Then $(\alpha'(t)\cdot u)^2 = \|\alpha'(t)\|^2 = (\alpha'(t)\cdot u)^2 + \|Y(t)\|$

hence $\|Y(t)\| = 0$ for all $t \Rightarrow Y = 0$.

Thus $\alpha'(t)$ is in the direction of $u \Rightarrow$

$\alpha'(t) = g(t)u$ and so $\alpha(t) = h(t)u + c$ where $h'(t) = g(t)$. Letting $h_1(t) = (h(t) - h(0))/\|q-p\|$ and $c_1 = h(0)u + c$

we see that $h(t) = h_1(t)\cdot\|q-p\| + h(0)$ and $c = c_1 - h(0)u \Rightarrow$

$$\alpha(t) = h(t)u + c = (h_1(t)\|q-p\| + h(0))\frac{q-p}{\|q-p\|} + c_1 - h(0)\frac{q-p}{\|q-p\|}$$

$$= h_1(t)(q-p) + c_1.$$

Since $\alpha(0) = p$ and $h_1(0) = h(0) - h(0) = 0$, $c_1 = p$.

Thus $\alpha(t) = h_1(t)(q-p) + p$

$$= (1 - h_1(t))p + h_1(t)q$$

$$= \sigma(h_1(t)),$$

and hence α is a reparametrization of σ, the straight line.

Therefore α is a straight line.

Also <u>note</u>: that $\|\alpha'(t)\| = \alpha'(t)\cdot u$ from above hence

$|h'(t)| = \|h'(t)u\| = h'(t)u\cdot u = h'(t)$ so $h'(t) \geq 0$

and hence $h_1'(t) = h'(t)/\|q-p\| \geq 0$