Solutions Homework #2 - Math 401 - Fall 2005

(1) p.27 #7:

(a) Yes, $\phi = dx_1 - dx_3$

(b) No, suppose $\phi(v_p) = p_1 - p_3$. Let $p = (1,0,0)$

then $\phi(v_p + w_p) = 1$

$\phi(v_p) + \phi(w_p) = 1 + 1 = 2$ $\Rightarrow$ ϕ is not linear on $T_p(\mathbb{R}^3)$

so not a 1-form

(c) Yes, $\phi = z\,dx + x\,dy$

(d) Yes, By Lemma 3.2 $V_p[f] = df(V_p)$

$\Rightarrow \phi = df$ where $f = x^2 + y^2$

$= 2x\,dx + 2y\,dy$.

(e) Yes, $\phi = 0$

(f) No, suppose $\phi(v_p) = (p_1)^2$. Let $p = (1,0,0)$ then

$\phi(v_p + w_p) = \phi((v+w)_p) = 1$ and

$\phi(v_p) + \phi(w_p) = 1 + 1 = 2$ so ϕ is not linear.

(2) p.32 #3

Let f be a differential function then

$$
\begin{aligned}
d(df) &= d\Big(\sum_i \partial f/\partial x_i \, dx_i\Big) \\
&= \sum_i d(\partial f/\partial x_i) \wedge dx_i \\
&= \sum_i \sum_j \partial^2 f/\partial x_j \partial x_i \, dx_j \wedge dx_i
\end{aligned}
$$

but this is 0 since $dx_i \wedge dx_i = 0$, $dx_j \wedge dx_i = -dx_i \wedge dx_j$ and $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (mixed partials are equal).

Hence by Theorem 6.4,

$$d(f\,dg) = df \wedge dg + f\, d(\cancel{dg})^{=0}$$
$$= df \wedge dg.$$

(3) p.32 #7

Let ϕ be a 1-form then $\phi = f dx + g dy + h dz$ hence

$$d(d\phi) = d(df \wedge dx + dg \wedge dy + dh \wedge dz)$$
$$= d\left(\frac{\partial f}{\partial y} dy\, dx + \frac{\partial f}{\partial z} dz\, dx + \frac{\partial g}{\partial x} dx\, dy + \frac{\partial g}{\partial z} dz\, dy + \frac{\partial h}{\partial x} dx\, dz + \frac{\partial h}{\partial y} dy\, dz\right)$$
$$= d\left[\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\, dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\, dx\right]$$

To compute $d\left[\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy\right]$ we only need to take the partial of $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$ w.r.t. z since $dx\, dx\, dy = dy\, dx\, dy = 0$.

Hence

$$d\left[\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy\right] = \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dz\, dx\, dy$$
$$= \left(\frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}\right) dx\, dy\, dz$$
$$= \left(\frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial y \partial z}\right) dx\, dy\, dz$$

Similarly

$$d\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dy\,dz\right]=\left(\frac{\partial^2 h}{\partial x\,\partial y}-\frac{\partial^2 g}{\partial z\,\partial x}\right)dx\,dy\,dz$$

and

$$d\left[\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dz\,dx\right]=\left(\frac{\partial^2 f}{\partial y\,\partial z}-\frac{\partial^2 h}{\partial x\,\partial y}\right)dx\,dy\,dz$$

hence $d(d\phi)=0$.

(4) p. 32 #8

(a) $\operatorname{grad} f=\sum\frac{\partial f}{\partial x_i}U_i \xleftrightarrow{(1)} \sum\frac{\partial f}{\partial x_i}dx_i = df$

(b) Suppose $\phi=\sum f_i\,dx_i$ and $\phi\xleftrightarrow{(1)}V$ then $V=\sum f_i U_i$.

Hence

$$d\phi=\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1dx_2+\left(\frac{\partial f_3}{\partial x_1}-\frac{\partial f_1}{\partial x_3}\right)dx_1dx_3+\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)dx_2dx_3$$

$$\xleftrightarrow{(2)}\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)U_3-\left(\frac{\partial f_3}{\partial x_1}-\frac{\partial f_1}{\partial x_3}\right)U_2+\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)U_1$$

$$=\operatorname{curl} V$$

(c) Let $\eta=f_1\,dx_2dx_3-f_2\,dx_1dx_3+f_3\,dx_1dx_2$ and $V\xleftrightarrow{(2)}\eta$

then $V=\sum f_iU_i$. Hence

$$d\eta=df_1\wedge dx_2dx_3-df_2\wedge dx_1dx_3+\partial f_3\wedge dx_1dx_2$$

$$=\frac{\partial f}{\partial x_1}dx_1dx_2dx_3-\frac{df}{\partial x_2}dx_2dx_1dx_3+\frac{\partial f}{\partial x_3}dx_3dx_1dx_2$$

$$=\left(\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial x_3}\right)dx_1dx_2dx_3=(\operatorname{div}V)\,dx_1dx_2dx_3$$

(5) Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a regular curve that does not satisfy $\alpha(t) = 0$ for any $t \in I$.

Let $t_0 \in I$ such that if $t \in I$ then $\|\alpha(t)\| \geq \|\alpha(t_0)\|$.

Thus the function $f(t) = \|\alpha(t)\|^2$ has a minimum at $t = t_0$ and hence $f'(t_0) = 0$. But

$$f'(t_0) = 2\,\alpha'(t_0) \cdot \alpha(t_0)$$

so $\alpha'(t_0) \cdot \alpha(t_0) = 0$ and $\alpha'(t_0)$ is orthogonal to $\alpha(t_0)$.