J. London Math. Soc. (2) 78 (2008) 677–692
doi:10.1112/jlms/jdn046

Homology and derived p-series of groups

Tim Cochran and Shelly Harvey

Abstract

We give homological conditions that ensure that a group homomorphism induces an isomorphism modulo any term of the *derived p-series*, in analogy to Stallings's 1963 result for the p-lower central series. In fact, we prove a stronger theorem that is analogous to Dwyer's extensions of Stallings' results. It follows that spaces that are $\mathbb{Z}_p$-homology equivalent have isomorphic fundamental groups modulo any term of their p-derived series. Various authors have related the ranks of the successive quotients of the p-lower central series and of the derived p-series of the fundamental group of a 3-manifold M to the volume of M, to whether certain subgroups of $\pi_1(M)$ are free, to whether finite index subgroups of $\pi_1(M)$ map onto non-abelian free groups, and to whether finite covers of M are 'large' in various other senses.

1. *Introduction*

In 1965, Stallings showed that if a group homomorphism induces an isomorphism on the first homology and an epimorphism on the second homology (with coefficients in $R = \mathbb{Z}$, $R = \mathbb{Q}$, or $R = \mathbb{Z}_p$), then it induces an isomorphism between the groups modulo any term of their respective *R-lower central series* (see [**31**]; see also [**18**, Theorem 9.11]). This was generalized in significant ways by Bousfield and Dwyer [**2**, **13**]. These results have had a significant impact in topology. In the present paper, we establish the analogs of Stallings' theorem and Dwyer's theorem for the *derived p-series*, that is to say, the analog for the derived series in the case $R = \mathbb{Z}_p$. We note that for $R = \mathbb{Z}$ and $R = \mathbb{Q}$, the precise analogs of Stallings' theorems for the *derived series* are easily seen to be false. In 2004, the second author introduced a new series, the *torsion-free derived series*, with which the authors proved analogs of the theorems of Stallings and Dwyer for $R = \mathbb{Q}$ (see [**4**, **5**, **17**]). In contrast, the case $R = \mathbb{Z}_p$ does not exhibit such complications. Since the quotient of a finitely generated group by any term of its derived p-series is a finite p-group (see Section 2), it is nilpotent, and for this reason the derived p-series behaves more like the p-lower central series.

The case $R = \mathbb{Z}_p$ is especially important in the study of 3-manifolds whose fundamental group controls most of their topology and where questions about the behavior of the homology of finite covers abound. Various authors have related the ranks of the successive quotients of the p-lower central series and of the derived p-series to the volumes of 3-manifolds [**12**], whether certain subgroups of π_1 of a 3-manifold are free [**4**, **30**], whether finite index subgroups of $\pi_1(M)$ map onto non-abelian free groups [**16**, **25**–**28**], and whether or not finite covers of M are 'large' in various other senses.

Recall that the *lower central series* of G, denoted by $\{G_n\}$, is inductively defined by

$$G_1 = G, \quad G_{n+1} = [G_n, G].$$

Fix a prime p. Our convention is that $\mathbb{Z}_p$ denotes the integers modulo p. The *p-lower central series* $\{G_{p,n}\}$ is the fastest descending central series with successive quotients that are $\mathbb{Z}_p$-vector

Received 10 April 2007; revised 15 April 2008; published online 30 July 2008.

2000 *Mathematics Subject Classification* 57M27, 57M07, 20J05 (primary), 55P60 (secondary).

The first author was partially supported by the National Science Foundation DMS-0406573,DMS-0706929, and the second author was partially supported by the Sloan Foundation and by NSF DMS-0539044.

spaces [**31**]. It is defined by

$$G_{p,1} = G, \quad G_{p,n+1} = (G_{p,n})^p [G_{p,n}, G].$$

Recall Stallings' result.

THEOREM 1.1 (Stallings' $\mathbb{Z}_p$ theorem [**31**, Theorem 3.4]). *Let $\phi : A \to B$ be a homomorphism that induces an isomorphism on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$. Then, for each n, ϕ induces an isomorphism $A/A_{p,n} \cong B/B_{p,n}$.*

Consequently, as was shown by Bousfield, the following corollary holds.

COROLLARY 1.2 [**2**, Lemma 3.7, Corollary 3.15, Remark 12.2]. *Under the hypotheses of Theorem 1.1, if $\dim\ H_1(A;\mathbb{Z}_p)$ is finite, then ϕ induces an isomorphism between the pro-p-completions $\hat{A} \to \hat{B}$.*

The *derived p-series* (sometimes called the p-derived series) denoted by $\{G^{(n)}\}$ is defined by

$$G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}](G^{(n)})^p.$$

(*Warning: the p will be suppressed throughout this paper.*) It is the fastest descending normal series with successive quotients that are $\mathbb{Z}_p$-vector spaces. We also set

$$G^{(\omega)} = \bigcap_{n=1}^{\infty} G^{(n)}.$$

Our first result is then the following corollary.

COROLLARY 4.3. *Let A and B be finitely generated groups. If $\phi : A \to B$ induces an isomorphism (respectively, monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$, then for each finite n, it induces an isomorphism (respectively, monomorphism) $A/A^{(n)} \to B/B^{(n)}$, and a monomorphism $A/A^{(\omega)} \subset B/B^{(\omega)}$.*

Actually, we show that this theorem can be derived easily from Stallings's theorem for the p-lower central series. Here it is also obtained as a corollary of our main theorem (Theorem 4.2 below), which is the analog of Dwyer's extensions of Stallings's theorems.

We have the standard applications.

COROLLARY 4.5. *Suppose that Y and X are path-connected CW-complexes with $\pi_1(Y)$ and $\pi_1(X)$ finitely generated. If $f : Y \to X$ is a continuous map that induces an isomorphism (respectively, monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$, then for each finite n, it induces an isomorphism (respectively, monomorphism)*

$$f_* : \pi_1(Y)/\pi_1(Y)^{(n)} \longrightarrow \pi_1(X)/\pi_1(X)^{(n)}.$$

COROLLARY 4.6. *If M and N are compact $\mathbb{Z}_p$-homology cobordant manifolds then for each n,*

$$\pi_1(M)/\pi_1(M)^{(n)} \cong \pi_1(N)/\pi_1(N)^{(n)}.$$

Note that the maps $f : N \to M$ between closed orientable 3-manifolds are particularly nice because if f_* is injective on $H_1(-;\mathbb{Z}_p)$, then it follows from the Poincaré duality that f_* is surjective on $H_2(-;\mathbb{Z}_p)$.

COROLLARY 4.7. *For any closed orientable* 3*-manifold N, there exists a hyperbolic* 3*-manifold M such that for each n,*

$$\pi_1(M)/\pi_1(M)^{(n)} \cong \pi_1(N)/\pi_1(N)^{(n)}$$

and

$$\pi_1(M)^{(n)}/\pi_1(M)^{(n+1)} \cong \pi_1(N)^{(n)}/\pi_1(N)^{(n+1)}.$$

In particular the growth rate of the successive quotients of the derived p-series of an arbitrary 3*-manifold is the same as that of a hyperbolic* 3*-manifold with the same integral homology. Moreover, $\pi_1(M)$ and $\pi_1(N)$ have isomorphic pro-p-completions.*

However, the last statement of this corollary could have been observed using Bousfield's result above.

These last corollaries are interesting in light of Lackenby's result.

THEOREM 1.3 [**27**, Theorem 1.12]. *Let G be a finitely presented group. If the successive quotients of the derived p-series have linear growth rate, then G is large; that is, G has a finite index subgroup that has a non-abelian free group as quotient.*

Here *linear growth rate* means

$$\inf(\dim_{\mathbb{Z}_p}(G^{(n)}/G^{(n+1)})/[G : G^{(n)}]) > 0.$$

In 1975, William Dwyer extended Stallings' lower-central series theorem for $R = \mathbb{Z}$ by weakening the hypothesis on H_2 and indeed found precise conditions for when a *specific* lower-central series quotient was preserved [**13**]. This was extended to the $R = \mathbb{Q}$ case by the authors in [**5**]. To motivate, for topologists, the philosophy of Dwyer's extension, consider that, for *spaces*, the hypothesis

$$H_2(\pi_1(Y);\mathbb{Z}) \longrightarrow H_2(\pi_1(X);\mathbb{Z})$$

is surjective, which is equivalent to saying that $H_2(X;\mathbb{Z})$ is generated by the classes from $H_2(Y;\mathbb{Z})$ together with all *spherical* classes, that is, classes that can be represented by maps of 2-spheres into X. Roughly speaking, Dwyer showed that if 'represented by 2-spheres' were replaced by 'represented by maps of surfaces into X which become spherical after killing $\pi_1(X)_m$', then Stallings' theorem still held for values of n roughly up to the fixed value m. A map $f : \Sigma \to X$ of a surface Σ becomes spherical after killing $\pi_1(X)_m$ if $H_1(\Sigma;\mathbb{Z})$ admits a symplectic basis $\{a_i, b_i \mid i = 1, \ldots, g\}$ of circles such that $f_*(a_i) \in \pi_1(X)_m$ for each i. Refer to [**5**] for a general discussion. Specifically, Dwyer defined, for any group B, an important subgroup

$$\Phi_m(B) \equiv \text{kernel}(H_2(B;\mathbb{Z}) \longrightarrow H_2(B/B_m;\mathbb{Z})) \subset H_2(B;\mathbb{Z}).$$

Dwyer's filtration of $H_2(B)$ has equivalent, more geometric, formulations in terms of *gropes*, certain 2-complexes which played a crucial role in Freedman and Teichner's work on 4-manifold topology that strengthened the results of Freedman and Quinn [**14**, **15**, **22**–**24**]. In actuality, Dwyer did not prove what we refer to as Dwyer's extensions of Stallings' theorems except in the case $R = \mathbb{Z}$. In particular, for the case $R = \mathbb{Z}_p$, he instead proved a version using group *cohomology* and defined a filtration of cohomology using the Massey products. Therefore, to complete our analogy and to fill this historical gap, we include in Section 3 (and just below) a statement and proof of a direct (*homological*) extension of Stallings' theorem for the lower-central p-series, that follows quite easily from mimicking Dwyer's proof for the $R = \mathbb{Z}$ case.

Specifically Dwyer's work suggests the following filtration of $H_2(B;\mathbb{Z}_p)$.

DEFINITION 1.4. Let $\Phi_{p,m}(B)$ denote the kernel of

$$H_2(B;\mathbb{Z}_p) \longrightarrow H_2(B/B_{p,m};\mathbb{Z}_p).$$

Note that, if $n \leqslant m$ then $\Phi_{p,m}(B) \subset \Phi_{p,n}(B)$.

Dwyer himself used a different filtration related to his cohomological approach. Armed with this definition we show the following theorem.

THEOREM 3.1. *If $\phi : A \to B$ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$ and an epimorphism $H_2(A;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)/\langle\Phi_{p,m}(B)\rangle$, then for any $n \leqslant m+1$, ϕ induces an isomorphism $A/A_{p,n} \to B/B_{p,n}$.*

To begin our attack on the derived p-series, we must first define a new filtration $\Phi^{(m)}(B)$ of $H_2(B;\mathbb{Z}_p)$ that is appropriate to the derived p-series:

$$\Phi^{(m)}(B) \equiv \text{image}(H_2(B^{(m)};\mathbb{Z}_p) \longrightarrow H_2(B;\mathbb{Z}_p)).$$

The more obvious analog, namely

$$\text{kernel}(H_2(B;\mathbb{Z}_p) \longrightarrow H_2(B/B^{(m)};\mathbb{Z}_p)),$$

turns out to be incorrect. It is difficult to explain at an intuitive level, why the more obvious analog is not the right one. For low-dimensional topologists, the following statement might have some resonance. The subgroup $\Phi_{p,m}(B)$ consists (loosely) of those homology classes that can be 'represented' by maps into $K(B,1)$ of *half-gropes* of class m. Classes that can be represented by maps of *full gropes* (symmetric gropes) of height m correspond to $\Phi^{(m)}(B)$ (see [**11**]).

Armed with this filtration, we can state our main theorem as follows.

THEOREM 4.2. *Let A be a finitely generated group and B a finitely presented group. If $\phi : A \to B$ induces an isomorphism (respectively, monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism $H_2(A;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)/\Phi^{(m)}(B)$, then for any $n \leqslant m+1$, ϕ induces an isomorphism (respectively, monomorphism) $A/A^{(n)} \to B/B^{(n)}$.*

The following corollary is a generalization of that of Stallings [**31**, Theorem 6.5]. It should be compared to Dwyer's result [**13**, Proposition 4.3], the hypothesis of which is in terms of the Massey products and the conclusion in terms of the *restricted mod p-lower central series*.

COROLLARY 4.8. *Let B be a finitely presented group and p a prime such that*

$$H_2(B^{(n-1)};\mathbb{Z}) \longrightarrow H_2(B;\mathbb{Z}_p)$$

is surjective. Let $\{x_i\}$ be a finite set of elements of B which is linearly independent in $H_1(B;\mathbb{Z}_p)$. Then the subgroup A generated by $\{x_i\}$ has the free p-solvable group $F/F^{(n)}$ as quotient (where F is free on $\{x_i\}$).

Proof of Corollary 4.8. Consider the induced map $\phi : F \to B$ and observe that it satisfies the hypotheses of Theorem 4.2. Thus

$$F/F(n) \longrightarrow B/B^{(n)}$$

is injective and factors through $A/A^{(n)}$. Thus $F/F^{(n)} \cong A/A^{(n)}$. $\square$

Also, in Section 5 we extend Stallings' application to link concordance to a more general equivalence relations on links.

2. *Basics*

We collect a few elementary observations that we need in the next section. No originality is claimed.

The derived p-series subgroups are verbal, and hence functorial and, in particular, fully invariant [**32**, p. 3]. It follows directly from the definition that $A^{(n)}/A^{(n+1)}$ is an abelian group, every element of which has order dividing p. Next we note that the derived p-series has an equivalent formulation.

Lemma 2.1. $A^{(n+1)} = \ker(A^{(n)} \xrightarrow{\pi\otimes 1} (A^{(n)}/[A^{(n)}, A^{(n)}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

Proof. Clearly

$$A^{(n+1)} = [A^{(n)}, A^{(n)}](A^{(n)})^p \subset \ker(\pi \otimes 1).$$

On the other hand, if we tensor the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot\ p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

with $C = A^{(n)}/[A^{(n)}, A^{(n)}]$, then we obtain the exact sequence

$$C \xrightarrow{\cdot\ p} C \xrightarrow{\mathrm{id}\otimes 1} C \otimes \mathbb{Z}_p \longrightarrow 0,$$

so the kernel of $\mathrm{id} \otimes 1$ is C^p. Then, since $\pi \otimes 1$ factors as

$$A^{(n)} \xrightarrow{\pi} A^{(n)}/[A^{(n)}, A^{(n)}] \xrightarrow{\mathrm{id}\otimes 1} A^{(n)}/[A^{(n)}, A^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

we see that the kernel of $\pi \otimes 1$ is $\pi^{-1}(C^p)$, which is clearly $[A^{(n)}, A^{(n)}](A^{(n)})^p$. $\square$

Lemma 2.2. *If A is finitely generated, then $A/A^{(n)}$ is a finite p-group with each of its elements having order dividing p^n.*

Proof. Proceed by induction on n. If $n = 0$, then $A/A^{(0)} = 1$ so the result holds. Suppose that $A/A^{(n)}$ is a finite p-group with each of its elements having order dividing p^n. Then $A^{(n)}$ is a subgroup of finite index of the finitely generated group A, and hence is itself finitely generated. Thus $A^{(n)}/A^{(n+1)}$ is a *finitely generated* abelian group with each of its elements having order dividing p, and hence is finite. Then from the exact sequence below we see that $A/A^{(n+1)}$ is an extension of a finite p-group by a finite p-group, and hence is itself a finite p-group and every element has order dividing p^{n+1}.

$$1 \longrightarrow \frac{A^{(n)}}{A^{(n+1)}} \longrightarrow \frac{A}{A^{(n+1)}} \longrightarrow \frac{A}{A^{(n)}} \longrightarrow 1. \qquad \square$$

A crucial observation is that the derived series is related to homology as follows.

Lemma 2.3. *$A^{(n)}/A^{(n+1)} \cong H_1(A; \mathbb{Z}_p[A/A^{(n)}])$ as right $\mathbb{Z}_p[A/A^{(n)}]$-modules.*

Proof. This can be seen in several ways. First consider the right-hand side of the equivalence. Note that the map $A \to A/A^{(n)}$ endows $\mathbb{Z}_p[A/A^{(n)}]$ with the structure of a $(\mathbb{Z}A, \mathbb{Z}_p[A/A^{(n)}])$-bimodule; similarly for $\mathbb{Z}[A/A^{(n)}]$. Thus

$$H_1(A; \mathbb{Z}_p[A/A^{(n)}]) \quad \text{and} \quad H_1(A; \mathbb{Z}[A/A^{(n)}])$$

are defined as right modules over $\mathbb{Z}_p[A/A^{(n)}]$ and $\mathbb{Z}[A/A^{(n)}]$, respectively. These modules have well-known interpretations as

$$H_1(A^{(n)}; \mathbb{Z}_p) \quad \text{and} \quad H_1(A^{(n)}; \mathbb{Z}),$$

respectively. A topologist usually thinks of the group A as being replaced by the Eilenberg–Maclane space $K(A, 1)$. From this perspective, the homology groups with twisted coefficients

$$H_1(A; \mathbb{Z}_p[A/A^{(n)}]) \quad \text{and} \quad H_1(A; \mathbb{Z}[A/A^{(n)}])$$

can be interpreted as merely the homology groups of the covering space, $K(A^{(n)}, 1)$, of $K(A, 1)$ with fundamental group $A^{(n)}$:

$$H_1(A^{(n)}; \mathbb{Z}_p) \quad \text{and} \quad H_1(A^{(n)}; \mathbb{Z}).$$

Here the group of covering translations is $A/A^{(n)}$, enabling us to view these as right modules over $\mathbb{Z}_p[A/A^{(n)}]$ or $\mathbb{Z}[A/A^{(n)}]$, respectively. Since the first homology is merely the abelianization, these homology modules have a purely group-theoretic description as

$$A^{(n)}/[A^{(n)}, A^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{and} \quad A^{(n)}/[A^{(n)}, A^{(n)}],$$

respectively. On the other hand, starting from the purely group-theoretic standpoint, $A^{(n)}/[A^{(n)}, A^{(n)}]$ is an abelian group on which A acts by conjugation ($x \to a^{-1}xa$) and $A^{(n)}$ acts trivially. Thus $A^{(n)}/[A^{(n)}, A^{(n)}]$ has the structure of a right $\mathbb{Z}[A/A^{(n)}]$-module. Similarly, $A^{(n)}/[A^{(n)}, A^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a $\mathbb{Z}_p$-vector space on which A acts by conjugation, giving it the structure of a right $\mathbb{Z}_p[A/A^{(n)}]$-module. The point is that it is well known that these module structures are the same as those derived just above.

Using the algebraist's view of group homology, the fact that

$$A^{(n)}/[A^{(n)}, A^{(n)}] \cong H_1(A; \mathbb{Z}[A/A^{(n)}])$$

is a consequence of Shapiro's lemma (or of the definition $H_1(A; \mathbb{Z}[A/A^{(n)}]) \equiv \mathrm{Tor}_1^A(\mathbb{Z}[A/A^{(n)}], \mathbb{Z})$ and the easy observation that the latter is $\mathrm{Tor}_1^{A^{(n)}}(\mathbb{Z}, \mathbb{Z}) \cong A^{(n)}/[A^{(n)}, A^{(n)}]$ (see [**18**, Lemma 6.2]).

Now, since $\pi \otimes 1$ above is surjective with kernel $A^{(n+1)}$, it induces a group isomorphism

$$A^{(n)}/A^{(n+1)} \xrightarrow{\pi\otimes 1} A^{(n)}/[A^{(n)}, A^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

which endows $A^{(n)}/A^{(n+1)}$ with the structure of a right $\mathbb{Z}_p[A/A^{(n)}]$-module. These remarks complete the proof. □

3. *Dwyer's theorem for the p-lower central series*

First, for historical completeness and to complete the analogy, we state and prove an improvement of Stallings' result for the *p-lower central series* that is suggested by Dwyer's work (although neither stated nor proved in this form by Dwyer). Dwyer proved a version for cohomology that employs a different p-series and uses a filtration defined in terms of the Massey products [**13**, Theorem 3.1]. It would be interesting to know how his result compares to that below.

Recall the filtration $\Phi_{p,m}(B)$ of $H_2(B;\mathbb{Z}_p)$ from Definition 1.4.

THEOREM 3.1. *If $\phi : A \to B$ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$ and an epimorphism $H_2(A;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)/\langle\Phi_{p,m}(B)\rangle$, then for any $n \leqslant m+1$, ϕ induces an isomorphism $A/A_{p,n} \to B/B_{p,n}$.*

Proof. We view m as fixed and proceed by induction on n. The case $n = 2$ is clear since $A/A_{p,2}$ is merely $H_1(A;\mathbb{Z}_p)$ and by hypothesis ϕ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$. Now assume that the theorem holds for $n \leqslant m$; that is, ϕ induces an isomorphism $A/A^{(n)} \cong B/B^{(n)}$. We will prove that it holds for $n+1$. Since the p-lower central series is fully invariant, the diagram below exists and is commutative.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \frac{A_{p,n}}{A_{p,n+1}} & \longrightarrow & \frac{A}{A_{p,n+1}} & \longrightarrow & \frac{A}{A_{p,n}} & \longrightarrow & 1 \\
 & & \big\downarrow \phi_n & & \big\downarrow \phi_{n+1} & & \big\downarrow \phi & & \\
1 & \longrightarrow & \frac{B_{p,n}}{B_{p,n+1}} & \longrightarrow & \frac{B}{B_{p,n+1}} & \longrightarrow & \frac{B}{B_{p,n}} & \longrightarrow & 1
\end{array}$$

In light of the five lemma, it suffices to show that ϕ_n is an isomorphism.

Now consider Stallings' exact sequence [**31**, Theorem 2.1], where all homology groups have $\mathbb{Z}_p$-coefficients.

$$\begin{array}{ccccccccc}
\longrightarrow & H_2(A) & \xrightarrow{\pi_A} & H_2(A/A_{p,n}) & \xrightarrow{\partial_A} & A_{p,n}/A_{p,n+1} & \longrightarrow & 0 \\
 & \big\downarrow \phi_* & & \big\downarrow (\phi_n)_* & & \big\downarrow \phi_n & & \\
\longrightarrow & H_2(B) & \xrightarrow{\pi_B} & H_2(B/B_{p,n}) & \xrightarrow{\partial_B} & B_{p,n}/B_{p,n+1} & \longrightarrow & 0
\end{array}$$

By our inductive hypothesis, $(\phi_n)_*$ is an isomorphism. It follows immediately that ϕ_n is surjective. Suppose that $\phi_n(a) = 0$. Choose α such that $\partial_A(\alpha) = a$. Then $(\phi_n)_*(\alpha) = \beta$, where $\partial_B(\beta) = 0$ so $\beta = \pi_B(\gamma)$. Since, $n \leqslant m$, by our hypothesis $\gamma = \phi_*(\delta) + \epsilon$ for some $\delta \in H_2(A)$ and ϵ in the kernel of π_B. It follows that $\pi_A(\delta) = \alpha$. Thus $a = \partial_A(\pi_A(\delta)) = 0$, and so ϕ_n is injective. □

4. *The derived p-series version of Stallings' and Dwyer's theorems*

Now we move on to the derived p-series and our main results. First we need to define the correct analog, for the derived p-series, of Dwyer's filtration of the second homology.

DEFINITION 4.1. Suppose that N is a normal subgroup of a group B. Let $\Phi^N(B)$ be the image of the inclusion-induced $H_2(N;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)$. Specifically, if $N = B^{(m)}$ then we abbreviate $\Phi^N(B)$ by $\Phi^{(m)}(B)$. Equivalently, $\Phi^N(B)$ consists of those classes represented by what we call *N-surfaces* of B. An N-surface of B is a continuous map $f : \Sigma \to K(B,1)$ of a compact oriented surface Σ, where $f_{|\partial\Sigma}$ factors through the standard map $z \to z^p$ on each circle (so Σ represents a mod p 2-cycle) and such that $f_*(\pi_1(\Sigma)) \subset N$.

THEOREM 4.2. *Let A be a finitely generated group and B a finitely presented group. If $\phi : A \to B$ induces an isomorphism (respectively, monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism $H_2(A;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)/\langle\Phi^{(m)}(B)\rangle$, then for any $n \leqslant m+1$, ϕ induces an isomorphism (respectively, monomorphism) $A/A^{(n)} \to B/B^{(n)}$.*

The following corollary is a precise analog, for the derived p-series, of Stallings' theorem for the p-lower central series. It can also be proved directly from Stallings' theorem and in that proof it is seen that B need only be finitely generated. A special case of this corollary is shown in [**6**, Theorem 1].

COROLLARY 4.3. *Let A and B be a finitely generated groups. If $\phi : A \to B$ induces an isomorphism (respectively, monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$, then for each finite n, it induces an isomorphism (respectively, monomorphism) $A/A^{(n)} \to B/B^{(n)}$. The map ϕ also induces a monomorphism $A/A^{(\omega)} \subset B/B^{(\omega)}$.*

Proof. In the case where B is finitely presented, this obviously follows from Theorem 4.2. This proof is independent of Stallings' theorem. What follows is a more direct proof that relies on Stallings' theorem.

Alternative Proof of Corollary 4.3. We consider only the case that ϕ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$. The monomorphism part follows by the same trick as is used later in the proof of Theorem 4.2. By Lemma 2.2, $A/A^{(n)}$ and $B/B^{(n)}$ are finite p-groups. It is well known that any finite p-group is nilpotent. Moreover, there is a $k > 0$ such that $(A/A^{(n)})_{p,k} = (B/B^{(n)})_{p,k} = 1$ [**31**, Lemma 4.2]. Thus, since the p-lower central series is fully invariant, the map $A \to A/A^{(n)}$ factors through $A/A_{p,k}$ and so $A_{p,k} \subset A^{(n)}$ and $B_{p,k} \subset B^{(n)}$. In examining the diagram below, Stallings's theorem implies that ϕ_*^k is an isomorphism.

$$\begin{array}{ccc}
\frac{A}{A_{p,k}} & \xrightarrow{\phi_*^k} & \frac{B}{B_{p,k}} \\
\downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\
\frac{A}{A^{(n)}} & \xrightarrow{\phi} & \frac{B}{B^{(n)}}
\end{array}$$

Since the derived p-series is characteristic, ϕ induces an isomorphism

$$\frac{A/A_{p,k}}{(A/A_{p,k})^{(n)}} \xrightarrow{\phi} \frac{B/B_{p,k}}{(B/B_{p,k})^{(n)}}.$$

It remains only to verify that

$$\frac{A}{A^{(n)}} \cong \frac{A/A_{p,k}}{(A/A_{p,k})^{(n)}}$$

which follows from the lemma below.

LEMMA 4.4. *If the kernel of the epimorphism $A \xrightarrow{f} C$ is contained in $A^{(n)}$, then*

$$f^{(n)} : \frac{A}{A^{(n)}} \longrightarrow \frac{C}{C^{(n)}}$$

is an isomorphism.

Proof. Clearly $f^{(n)}$ is surjective. Suppose that $f^{(n)}([a]) = 0$, so $f(a) \in C^{(n)}$. Since f is surjective and the derived p-series is verbal, there is some $a' \in A^{(n)}$ such that $f(a') = f(a)$. Hence $a(a')^{-1}$ is in the kernel of f. Thus $a \in A(n)$. □

This completes the proof of Corollary 4.3. □

The proofs of the following corollaries are standard (compare [**31**]).

COROLLARY 4.5. *Suppose that Y and X are path-connected* CW*-complexes with $\pi_1(Y)$ and $\pi_1(X)$ finitely generated. If $f : Y \to X$ is a continuous map that induces an isomorphism* (*respectively monomorphism*) *on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$, then for each finite n, it induces an isomorphism* (*respectively, monomorphism*)

$$f_* : \pi_1(Y)/\pi_1(Y)^{(n)} \longrightarrow \pi_1(X)/\pi_1(X)^{(n)}.$$

Proof. Let $A = \pi_1(Y)$ and $B = \pi_1(X)$. Of course

$$H_1(Y;\mathbb{Z}_p) \cong H_1(A;\mathbb{Z}_p).$$

Since a $K(A,1)$ can be obtained from Y by adjoining only cells of dimension 3 and greater, the natural map

$$H_2(Y;\mathbb{Z}_p) \longrightarrow H_2(\pi_1(Y);\mathbb{Z}_p)$$

is surjective. Thus the hypothesis that

$$f_* : H_2(Y;\mathbb{Z}_p) \longrightarrow H_2(X;\mathbb{Z}_p)$$

is surjective implies that

$$f_* : H_2(A;\mathbb{Z}_p) \longrightarrow H_2(B;\mathbb{Z}_p)$$

is surjective. Now apply Corollary 4.3. □

COROLLARY 4.6. *If M and N are compact $\mathbb{Z}_p$-homology cobordant manifolds, then for each n,*

$$\pi_1(M)/\pi_1(M)^{(n)} \cong \pi_1(N)/\pi_1(N)^{(n)}.$$

Proof. Let C be the cobordism. Then the two inclusion maps induce $\mathbb{Z}_p$-homology isomorphisms so Corollary 4.5 and be applied to each. Thus

$$\pi_1(M)/\pi_1(M)^{(n)} \cong \pi_1(C)/\pi_1(C)^{(n)} \cong \pi_1(N)/\pi_1(N)^{(n)}. \qquad \square$$

COROLLARY 4.7. *For any closed orientable* 3*-manifold N, there exists a hyperbolic* 3*-manifold M such that for each n,*

$$\pi_1(M)/\pi_1(M)^{(n)} \cong \pi_1(N)/\pi_1(N)^{(n)}$$

and

$$\pi_1(M)^{(n)}/\pi_1(M)^{(n+1)} \cong \pi_1(N)^{(n)}/\pi_1(N)^{(n+1)}.$$

Hence the growth rate of the derived p-series of an arbitrary 3*-manifold is the same as that of a hyperbolic* 3*-manifold with the same integral homology. Moreover, $\pi_1(M)$ and $\pi_1(N)$ have isomorphic pro-p-completions.*

Proof. Most of this is an immediate consequence of Corollary 4.6 and the fact, due to Ruberman, that any closed orientable 3-manifold is homology cobordant to a hyperbolic 3-manifold [**29**, Theorem 2.6]. However, if one wants the degree 1 map, then it is easier to apply the later result of Kawauchi: given N, there exist a hyperbolic M and a degree 1 map $f : M \to N$ that induces an isomorphism on all homology groups [**20**, **21**] (rediscovered in [**1**]). Then simply apply Corollary 4.5. The equivalence of the pro-p-completions follows immediately from Bousfield's result, Corollary 1.2. □

The following is a generalization of Stallings' [**31**, Theorem 6.5]. It should be compared to Dwyer's result [**13**, Proposition 4.3], the hypothesis of which is in terms of the Massey products and the conclusion in terms of the *restricted mod p-lower central series*.

Corollary 4.8. *Let B be a finitely presented group and p a prime such that*

$$H_2(B^{(n-1)};\mathbb{Z}_p) \longrightarrow H_2(B;\mathbb{Z}_p)$$

is surjective ($H_2(B;\mathbb{Z}_p)$ is generated by $B^{(n-1)}$-surfaces). Let $\{x_i\}$ be a finite set of elements of B which is linearly independent in $H_1(B;\mathbb{Z}_p)$. Then the subgroup A generated by $\{x_i\}$ has the free p-solvable group $F/F^{(n)}$ as quotient (where F is free on $\{x_i\}$).

Proof. Consider the induced map $\phi : F \to B$ and observe that it satisfies the hypotheses of Theorem 4.2. Thus

$$F/F(n) \longrightarrow B/B^{(n)}$$

is injective and factors through $A/A^{(n)}$. Thus $F/F^{(n)} \cong A/A^{(n)}$. □

Proof of Theorem 4.2. We first consider the case that ϕ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$. We view m as fixed and proceed by induction on n. The case $n = 1$ is clear since $A/A^{(1)}$ is merely $H_1(A;\mathbb{Z}_p)$ by Lemma 2.3 and by hypothesis ϕ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$. Now assume that the first claim holds for $n \leqslant m$; that is, ϕ induces an isomorphism $A/A^{(n)} \cong B/B^{(n)}$. We will prove that it holds for $n+1$.

Since the derived p-series is fully invariant, $\phi(A^{(n+1)}) \subset B^{(n+1)}$. Hence the diagram below exists and is commutative. In light of the five lemma, it suffices to show that ϕ induces an isomorphism $A^{(n)}/A^{(n+1)} \to B^{(n)}/B^{(n+1)}$.

$$\begin{tikzcd}
1 \arrow[r] & \frac{A^{(n)}}{A^{(n+1)}} \arrow[r] \arrow[d, "\phi_n"] & \frac{A}{A^{(n+1)}} \arrow[r] \arrow[d, "\phi_{n+1}"] & \frac{A}{A^{(n)}} \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & \frac{B^{(n)}}{B^{(n+1)}} \arrow[r] & \frac{B}{B^{(n+1)}} \arrow[r] & \frac{B}{B^{(n)}} \arrow[r] & 1
\end{tikzcd}$$

Now Lemma 2.3 suggests that this problem can be translated into a homological one. In preparation, note that the inductive hypothesis is that ϕ induces an isomorphism $A/A^{(n)} \to B/B^{(n)}$ and hence a ring isomorphism $\phi : \mathbb{Z}_p[A/A^{(n)}] \to \mathbb{Z}_p[B/B^{(n)}]$. Thus any left or right $\mathbb{Z}_p[B/B^{(n)}]$-module inherits a left or right $\mathbb{Z}_p[A/A^{(n)}]$-module structure, respectively, via ϕ. The map ϕ also endows $\mathbb{Z}_p[B/B^{(n)}]$ with the structure of a $(\mathbb{Z}_p[A/A^{(n)}] - \mathbb{Z}_p[B/B^{(n)}])$-bimodule with respect to which ϕ is a map of $(\mathbb{Z}_p[A/A^{(n)}] - \mathbb{Z}_p[A/A^{(n)}])$-bimodules.

Now consider the following commutative diagram, where the horizontal equivalences follow from Lemma 2.3.

$$\begin{tikzcd}
H_1(A;\mathbb{Z}_p[A/A^{(n)}]) \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes\phi"] & \frac{A^{(n)}}{A^{(n+1)}} \\
H_1(A;\mathbb{Z}_p[B/B^{(n)}]) \arrow[d, "\phi\otimes\mathrm{id}"] & \\
H_1(B;\mathbb{Z}_p[B/B^{(n)}]) \arrow[r, "\cong"] & \frac{B^{(n)}}{B^{(n+1)}}
\end{tikzcd}$$

We claim that the composition $\phi_n = (\phi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\phi)$ is an isomorphism. Since $\phi : A/A^{(n)} \to B/B^{(n)}$ is an isomorphism, $(\mathrm{id}\otimes\phi)$ is clearly an isomorphism. Finally, we show that $\phi\otimes\mathrm{id}$ is an isomorphism using Proposition 4.9 below (setting $\Gamma = B/B^{(n)}, N = B^{(n)}$). Note that $B/B^{(n)}$ is a finite p-group by Lemma 2.2. This completes the proof of the 'isomorphism' case of Theorem 4.2, modulo the proof of Proposition 4.9.

PROPOSITION 4.9. *Suppose that A is finitely generated and B is finitely related. Suppose that $\phi : A \to B$ induces a monomorphism (respectively, an isomorphism) on $H_1(-;\mathbb{Z}_p)$. Consider the coefficient system $\psi : B \to \Gamma$, where Γ is a finite p-group. Suppose that $N \subset \ker\psi$ and suppose that*

$$H_2(A;\mathbb{Z}_p) \longrightarrow H_2(B;\mathbb{Z}_p)/\Phi^N(B)$$

is surjective. Equivalently, suppose that $H_2(B;\mathbb{Z}_p)$ is spanned by $\phi_*(H_2(A;\mathbb{Z}_p))$ together with a collection of N-surfaces. Then ϕ induces a monomorphism (respectively, an isomorphism)

$$\phi_* : H_1(A;\mathbb{Z}_p\Gamma) \longrightarrow H_1(B;\mathbb{Z}_p\Gamma).$$

Before proving Proposition 4.9, we finish the proof of Theorem 4.2. Suppose that we assume only that ϕ induces a *monomorphism* on $H_1(-;\mathbb{Z}_p)$. We are grateful to Kent Orr for suggesting the idea for this argument. Since $H_1(B;\mathbb{Z}_p)$ is a $\mathbb{Z}_p$-vector space, it decomposes as an image $(\phi) \oplus C$ for some vector space C with basis $\{c_i \mid i \in \mathcal{C}\}$. Let F be the free group on $\mathcal{C}$ and $\tilde{A} = A * F$. We can extend ϕ to a map $\psi : \tilde{A} \to B$, by setting $\psi(x_i) = c_i$. Observe that ψ induces an isomorphism on $H_1(-;\mathbb{Z}_p)$. Moreover, there is an obvious retraction $r : \tilde{A} \to A$, under which $H_2(\tilde{A};\mathbb{Z}_p)$ maps onto $H_2(A;\mathbb{Z}_p)$, so we recover our hypothesis on H_2 as well. Thus, by the first part of the theorem, for any finite n, ψ induces an isomorphism $\tilde{A}/\tilde{A}^{(n)} \to B/B^{(n)}$. Since the derived p-series is fully invariant, there is a retraction $r : \tilde{A}/\tilde{A}^{(n)} \to A/A^{(n)}$ showing that $A/A^{(n)} \to \tilde{A}/\tilde{A}^{(n)}$ is injective. Thus ϕ induces a monomorphism $A/A^{(n)} \to B/B^{(n)}$. This completes the proof of Theorem 4.2, modulo the proof of Proposition 4.9.

Proof of Proposition 4.9. We first remark that since ϕ_* is certainly a map of $\mathbb{Z}_{|}\Gamma$-modules, it suffices to show that ϕ_* induces a bijection of $\mathbb{Z}_p$-vector spaces. Then, since any finite p-group is poly-(cyclic of p-power order), it would suffice to prove the theorem for $\Gamma = \mathbb{Z}_{p^r}$. The general proof is not much more difficult.

We offer a 'topological' proof. We may consider that A, B are Eilenberg–Maclane spaces with a finite number of 1-cells for A and 2-cells for B and that ϕ is cellular. By replacing the chosen $K(B,1)$ with the mapping cylinder of ϕ, we may assume that $K(A,1)$ is a subcomplex of $K(B,1)$ and that the latter has a finite 2-skeleton. Thus with such a cell structure the ordinary cellular chain complex $\overline{\mathcal{C}}_* = \overline{C}_*(B,A;\mathbb{Z}_p)$ and the twisted relative cellular chain complex

$$\mathcal{C}_* \equiv \mathcal{C}_*(B,A;\mathbb{Z}_p\Gamma)$$

will be finitely generated in dimension 2. If one thinks of ψ as inducing a principal Γ-bundle B_Γ over B and $\psi \circ \phi$ as inducing one, A_Γ, over A, then $\mathcal{C}_*$ is merely the relative cellular chain complex for (B_Γ, A_Γ) with $\mathbb{Z}_p\Gamma$ coefficients. Let $\overline{\mathcal{C}}_* = \mathcal{C}_* \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$ with $\pi_\# : \mathcal{C}_* \to \overline{\mathcal{C}}_*$ and note that $\overline{\mathcal{C}}_*$ may be identified with the ordinary cellular chain complex of (B,A) with $\mathbb{Z}_p$-coefficients.

Let $\{\overline{\Sigma}_s \mid s \in S\}$ denote the collection of N-surfaces in B and let $\{\bar{x}_s\}$ denote the 2-cycles in $\overline{C}_2(B,A)$ represented by $\{\overline{\Sigma}_s\}$. Examining the exact sequence

$$H_2(A;\mathbb{Z}_p) \xrightarrow{\phi_*} H_2(B;\mathbb{Z}_p) \longrightarrow H_2(B,A;\mathbb{Z}_p) \xrightarrow{\partial_*} H_1(A;\mathbb{Z}_p) \xrightarrow{\phi_*} H_1(B;\mathbb{Z}_p),$$

where, by hypothesis, ϕ_* is injective on $H_1(-;\mathbb{Z}_p)$, one sees that our hypotheses are engineered precisely so that $H_2(\overline{\mathcal{C}}_*)$ is spanned by $\{[\bar{x}_s]\}$. Hence

$$\operatorname{rank}_{\mathbb{Z}_p}(H_2(\overline{\mathcal{C}}_*)/\langle\{[\bar{x}_s]\}\rangle) = 0.$$

Now since $N \subseteq \ker\psi$, the $\overline{\Sigma}_s$ are also $(\ker\psi)$-surfaces and so lift to B_Γ. Choose lifts $\{\Sigma_s \mid s \in S\}$. That is, by definition, the $\{\overline{\Sigma}_s\}$ can be lifted to $H_2(N;\mathbb{Z}_p)$ and hence can be lifted to represent classes $\{[\Sigma_s]\}$ in $H_2(B_\Gamma;\mathbb{Z}_p)\cong H_2(B;\mathbb{Z}_p\Gamma)$, and hence represent classes in $H_2(B,A;\mathbb{Z}_p\Gamma)$. Let $\{x_s\mid s\in S\}$ denote the 2-cycles of $C_2(B,A;\mathbb{Z}_p\Gamma)$ represented by $\{\Sigma_s\}$. Note that $\bar{x}_s=\pi_\#(x_s)$. Now consider the exact sequence

$$H_2(B;\mathbb{Z}_p\Gamma)\xrightarrow{\pi_*}H_2(B,A;\mathbb{Z}_p\Gamma)\xrightarrow{\partial_*}H_1(A;\mathbb{Z}_p\Gamma)\xrightarrow{\phi_*}H_1(B;\mathbb{Z}_p\Gamma).$$

Our aim, namely proving that ϕ_* is injective, is now seen to be equivalent to showing that π_* is surjective. Let $\langle\{[x_s]\}\rangle$ denote the $\mathbb{Z}_p\Gamma$-submodule generated by $\{[x_s]\}$. Since $\{[x_s]\}\subset\operatorname{image}\pi_*$, it suffices to show that

$$\operatorname{rank}_{\mathbb{Z}_p}(H_2(\mathcal{C}_*)/\left\langle\{[x_s]\}\right\rangle)=0.$$

Therefore, we are reduced to proving a homological statement:

$$\operatorname{rank}_{\mathbb{Z}_p}(H_2(\overline{\mathcal{C}}_*)/\left\langle\{[\bar{x}_s]\}\right\rangle)=0\quad\Longrightarrow\quad\operatorname{rank}_{\mathbb{Z}_p}(H_2(\mathcal{C}_*)/\left\langle\{[x_s]\}\right\rangle)=0.$$

We can simplify this even more as follows. We can define a projective chain complex $\mathcal{D}_*=\{D_q,d_q\}$ such that $H_2(\mathcal{D}_*)\cong H_2(\mathcal{C}_*)/\left\langle[x_s]\mid s\in S\right\rangle$. Set $D_3=(\bigoplus_{s\in S}R\Gamma)\bigoplus C_3$ and otherwise set $D_q=C_q$. Let $d_3:D_3\to D_2$ be defined by $d_3(e_s,y)=x_s+\partial_3(y)$, where $\{e_s\}$ is a basis of $(R\Gamma)^s$ and $y\in C_3$, and $d_4:D_4\to D_3$ by $d_4(z)=(0,\partial_4(z))$ for $z\in D_4=C_4$. Then the 2-cycles of $\mathcal{D}_*$ are the same as those of $\mathcal{C}_*$ while the group of 2-boundaries is larger (it includes $\{x_s\}$). Hence $H_2(\mathcal{D}_*)\cong H_2(\mathcal{C}_*)/\left\langle[x_s]\mid s\in S\right\rangle$ as claimed. Note that $D_2=C_2$ is finitely generated. Similarly we can define a chain complex $\overline{\mathcal{D}}_*$ which agrees with $\overline{\mathcal{C}}_*$ except in dimension 3, where $\overline{D}_3=(\bigoplus_{s\in S}R)\bigoplus\overline{\mathcal{C}}_*$ with $\bar{d}_3:\overline{D}_3\to\overline{D}_2$ given by $d_3(\bar{e}_s,\bar{y})=\bar{x}_s+\overline{\partial}_3(\bar{y})$ for $\{\bar{e}_s\}$ a basis of R^s and $\bar{y}\in\overline{C}_3$. Then, just as above:

$$H_2(\overline{\mathcal{D}}_*)\cong H_2(\overline{\mathcal{C}}_*)/\left\langle[x_s]\mid s\in S\right\rangle.$$

With this translation we are reduced to proving:

$$\operatorname{rank}_{\mathbb{Z}_p}(H_2(\overline{\mathcal{D}}_*)))=0\quad\Longrightarrow\quad\operatorname{rank}_{\mathbb{Z}_p}(H_2(\mathcal{D}_*)))=0.$$

Moreover, the chain map $\pi:\mathcal{C}_*\to\overline{\mathcal{C}}_*$ extends to $\tilde{\pi}:\mathcal{D}_*\to\overline{\mathcal{D}}_*$ if we set $\tilde{\pi}(e_s,y)=(\bar{e}_s,\pi(y))$, and one sees that $\overline{\mathcal{D}}_*=\mathcal{D}_*\otimes_{\mathbb{Z}_p\Gamma}\mathbb{Z}_p$; then one can apply Lemma 4.10 to this chain complex.

The desired result now follows immediately from Lemma 4.10 (proof postponed).

Lemma 4.10. *Suppose that Γ is a finite p-group and $\mathcal{D}_*$ is a projective right $\mathbb{Z}_p\Gamma$ chain complex with D_q finitely generated. Then*

$$\operatorname{rank}_{\mathbb{Z}_p}H_q(\mathcal{D}_*)\leqslant|\Gamma|\operatorname{rank}_{\mathbb{Z}_p}H_q(\mathcal{D}_*\otimes_{\mathbb{Z}_p\Gamma}\mathbb{Z}_p).$$

This concludes the proof that ϕ_* is *injective* with $\mathbb{Z}_p\Gamma$ coefficients modulo the proof of Lemma 4.10.

Before proving Lemma 4.10, consider the case that, in addition, ϕ_* is an *isomorphism* on $H_1(-;\mathbb{Z}_p)$. Since B is finitely generated and we can assume that $K(A,1)$ has only a finite number of 0 cells, after the mapping cylinder construction, it follows that $\mathcal{C}_*(B,A;\mathbb{Z}_p\Gamma)$ is finitely generated in dimension 1. Again, by Lemma 4.10, we see that

$$\operatorname{rank}_{\mathbb{Z}_p}H_1(\mathcal{C}_*)\leqslant|\Gamma|\operatorname{rank}_{\mathbb{Z}_p}H_1(\overline{\mathcal{C}}_*)$$

and thus

$$\operatorname{rank}_{\mathbb{Z}_p}H_1(B,A;\mathbb{Z}_p)\leqslant|\Gamma|\operatorname{rank}_{\mathbb{Z}_p}H_1(B,A;\mathbb{Z}_p)=0.$$

It follows that ϕ induces an epimorphism and hence an isomorphism

$$\phi_*:H_1(A;\mathbb{Z}_p\Gamma)\longrightarrow H_1(B;\mathbb{Z}_p\Gamma).$$

This completes the proof of Proposition 4.9 modulo the proof of Lemma 4.10. In order to prove Lemma 4.10 we will need the following result of R. Strebel and a modest corollary.

PROPOSITION 4.11 (Strebel [**33**, Lemma 1.10]). *Suppose that p is a prime integer and Γ is a finite p-group. Any map between projective right $\mathbb{Z}_p\Gamma$-modules whose image under the functor $-\otimes_{\mathbb{Z}_p\Gamma}\mathbb{Z}_p$ is injective is itself injective.*

REMARK. Strebel established this property for a much larger class of groups which Howie and Schneebli showed was precisely the class of all locally p-indicable groups [**19**].

LEMMA 4.12. *Suppose that $\tilde{f}: M \to N$ is a homomorphism between projective $\mathbb{Z}_p\Gamma$-modules with Γ a finite p-group, and let $f = \tilde{f} \otimes \mathrm{id}$ be the induced homomorphism of $\mathbb{Z}_p$-vector spaces $M \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p \to N \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$. Then $\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \tilde{f}) \geqslant |\Gamma| \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} f)$, where $|\Gamma|$ is the order of Γ.*

Proof. By the rank of a homomorphism we shall mean the rank of its image. Suppose that $\operatorname{rank}_{\mathbb{Z}_p} f \geqslant r$. Then there is a *monomorphism* $g: \mathbb{Z}_p^r \to N \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$ whose image is a subgroup of image f. If e_i, $1 \leqslant i \leqslant r$, is a basis of $\mathbb{Z}_p^r$, then choose $M_i \in M \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$ such that $f(M_i) = g(e_i)$. Since the 'augmentation' $\epsilon_M : M \to M \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$ is surjective there exist elements $m_i \in M$ such that $\epsilon_M(m_i) = M_i$. Consider the map $\tilde{g} : (\mathbb{Z}_p\Gamma)^r \to N$ defined by sending the ith basis element to $\tilde{f}(m_i)$. The augmentation of $\tilde{g}$, $\tilde{g} \otimes \mathrm{id}$, is the map $(\mathbb{Z}_p\Gamma)^r \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p \to N \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$ that sends e_i to $\epsilon_N(\tilde{f}(m_i)) = f(\epsilon_M(m_i)) = g(e_i)$ and thus is seen to be identifiable with g. In particular $\tilde{g} \otimes \mathrm{id}$ is a monomorphism, and thus by Proposition 4.11, $\tilde{g}$ is a monomorphism. Since the image of $\tilde{g}$ lies in the image of $\tilde{f}$, $\tilde{g}$ yields a monomorphism from $(\mathbb{Z}_p\Gamma)^r$ into the image of $\tilde{f}$, showing that the $\mathbb{Z}_p$-rank of image $\tilde{f}$ is at least r times the order of Γ. □

Proof of Lemma 4.10. Let $\{\overline{\mathcal{D}}_*\} = \{\overline{D}_q, \overline{\partial}_q\} = \{D_q \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p, \partial_q \otimes \mathrm{id}\}$, let $\overline{r}_q = \operatorname{rank}_{\mathbb{Z}_p} \overline{D}_q$ and let $r_q = \operatorname{rank}_{\mathbb{Z}_p} D_q$. Since D_q is finitely generated and projective and Γ is finite, we claim that $r_q = |\Gamma| \overline{r}_q$. This is obvious if D_q is free and requires a small argument if not (the same as that in the proof of [**5**, Corollary 2.8]).

Now observe that

$$\begin{aligned}
\operatorname{rank}_{\mathbb{Z}_p} H_q(\mathcal{D}_*) &= \operatorname{rank}_{\mathbb{Z}_p}(\ker \partial_q) - \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \partial_{q+1}) \\
&= r_q - \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \partial_q) - \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \partial_{q+1}) \\
&\leqslant |\Gamma| \overline{r}_q - |\Gamma| \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \overline{\partial}_q) - |\Gamma| \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \overline{\partial}_{q+1}) \\
&= |\Gamma| (\operatorname{rank}_{\mathbb{Z}_p}(\ker \overline{\partial}_q) - \operatorname{rank}_{\mathbb{Z}_p}(\operatorname{image} \overline{\partial}_{q+1})) \\
&= |\Gamma| \operatorname{rank}_{\mathbb{Z}_p} H_q(\overline{\mathcal{D}}_*),
\end{aligned}$$

where the inequality follows from two applications of Lemma 4.12 and our observation that $r_q = |\Gamma| \overline{r}_q$. This completes the proof of Lemma 4.10. □

Thus the proof of Proposition 4.9, and hence that of Theorem 4.2, are complete. □

The following is a useful consequence.

COROLLARY 4.13. *Suppose that Γ is a finite p-group.*

(a) *If C_* is a non-negative right $\mathbb{Z}_p\Gamma$ chain complex that is finitely generated and projective in dimensions $0 \leqslant i \leqslant m$ such that $H_i(C_* \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p) = 0$ for $0 \leqslant i \leqslant m$, then $H_i(C_*) = 0$ for $0 \leqslant i \leqslant m$.*

(b) *If $f : Y \to X$ is a continuous map between connected* CW *complexes, X having a finite m-skeleton and Y a finite $(m-1)$-skeleton, which is m-connected on $\mathbb{Z}_p$ homology, and $\phi : \pi_1(X) \to \Gamma$ is a coefficient system, then f is m-connected on homology with $\mathbb{Z}_p\Gamma$-coefficients. In particular, if Y_Γ and X_Γ denote the induced Γ-covering spaces of Y and X then any lift $\tilde{f} : Y_\Gamma \to X_\Gamma$ of f is m-connected on $\mathbb{Z}_p$ homology.*

Proof. Let $\epsilon : \mathbb{Z}_p\Gamma \to \mathbb{Z}_p$ be the augmentation and let $\epsilon(C_*)$ denote $C_* \otimes_{\mathbb{Z}_p\Gamma} \mathbb{Z}_p$. Since $\epsilon(C_*)$ is acyclic up to dimension m, there is a 'partial' chain homotopy

$$\{h_i : \epsilon(C_*)_i \longrightarrow \epsilon(C_*)_{i+1} \mid 0 \leqslant i \leqslant m\}$$

between the identity and the zero chain homomorphisms. By this we mean that

$$\partial h_i + h_{i-1}\partial = \text{id} \quad \text{for } 0 \leqslant i \leqslant m.$$

Since $C_i \xrightarrow{\epsilon} \epsilon(C_i)$ is surjective and C_i is projective, $h_i \circ \epsilon : C_i \to \epsilon(C_i)$ can be lifted to $\widetilde{h}_i : C_i \to C_i$ so that $\epsilon \circ \widetilde{h}_i = h_i \circ \epsilon$. In this manner, h can be lifted to a partial chain homotopy $\{\widetilde{h}_i \mid 0 \leqslant i \leqslant m\}$ on C_* between some partial chain map $\{f_i \mid 0 \leqslant i \leqslant m\}$ and the zero map. Moreover, $\epsilon(f_i)$ is the identity map on $\epsilon(C_*)_i$, and in particular, is injective. Thus, by Proposition 4.11, $f_i : C_i \to C_i$ is an injective map of $\mathbb{Z}_p\Gamma$-modules for each i. Note that C_i is also a $\mathbb{Z}_p$-vector space. Since C_i is a finitely generated projective $\mathbb{Z}_p\Gamma$-module, it is a direct summand of a finitely generated free $\mathbb{Z}_p\Gamma$-module. Since Γ is a finite group, $\mathbb{Z}_p\Gamma$ is a finite-dimensional $\mathbb{Z}_p$-vector space. It follows that C_i is a *finite-dimensional* $\mathbb{Z}_p$-vector space. Any injective map between finite-dimensional vector spaces of the same rank is necessarily bijective. Thus f_i is bijective. Thus f_i is a bijective morphism of $\mathbb{Z}_p\Gamma$-modules, and hence an isomorphism of $\mathbb{Z}_p\Gamma$-modules for each i. Consequently, $\widetilde{h}_i$ is a partial chain homotopy on C_* between the zero map and the partial chain map f_i. To see that C_* is acyclic, suppose that $z \in C_i$ is a cycle, $0 \leqslant i \leqslant m$. Then $f_i(w) = z$ for some $w \in C_i$. Thus

$$\partial h_i(w) + h_{i-1}\partial(w) = f_i(w) = z.$$

If w is a cycle then we see immediately that z is a boundary. To see that w is a cycle, apply boundary to both sides of the above equation to yield $\partial h_{i-1}\partial(w) = 0$. Hence

$$0 = (\partial h_{i-1})\partial(w) = (f_{i-1} - h_{i-2}\partial)\partial(w) = f_{i-1}(\partial w)$$

which implies that $\partial(w) = 0$ since f_{i-1} is an isomorphism. Hence we have shown that C_* is acyclic up to and including dimension m.

The second statement follows from applying this to the relative cellular chain complex associated to the mapping cylinder of f. □

5. *Topological applications*

Stallings' and Dwyer's theorems have been instrumental in the study of homology cobordism of 3-manifolds and, in particular, in the study of link concordance. Our current results may be similarly applied. By a *link* L of m components we mean an oriented, ordered collection of m circles disjointly and smoothly embedded in S^3. Two links L_0 and L_1 are *concordant* if there exist m annuli disjointly embedded in $S^3 \times [0,1]$, restricting to yield L_j on $S^3 \times \{j\}$, $j = 0, 1$. The complement of the union of annuli is easily seen, by Alexander duality, to be a product on integral homology, and thus the fundamental groups of the link complements are related by Stallings' theorems. Recently, several other weaker equivalence relations on knots and links have been considered and found to be useful in understanding knot and link concordance [**7**, **8**, **9**, **10**, **11**, **17**, **23**, **24**, **34**]. These equivalence relations involve replacing the annuli in the definition of concordance by surfaces equipped with some extra structure depending on a parameter. As the parameter increases, these surfaces are to be viewed as 'approximating' annuli, and hence yielding 'filtrations' of link concordance. Below we show that our results generalize Stallings' results on link concordance to some of these more general equivalence relations.

Recall that Stallings showed that concordant links have exteriors with fundamental groups that are isomorphic modulo any term of the lower central p-series. We can prove an analog for the derived p-series and moreover use our Dwyer-type theorem to generalize his result to the following equivalence relation, which is weaker than concordance.

Definition 5.1. The m-component links L_0 and L_1 are *(n)-$\mathbb{Z}_p$-cobordant* if there exist compact oriented surfaces $\Sigma_i, 1 \leqslant i \leqslant m$, properly and disjointly embedded in $S^3 \times [0,1]$, restricting to yield L_j on $S^3 \times \{j\}$, $j = 0, 1$, such that the image of each $\pi_1(\Sigma_i^+)$ in

$$\pi_1\Big((S^3 \times [0,1]) - \coprod \Sigma_i\Big) \equiv \pi_1(E)$$

is contained in $\pi_1(E)^{(n)}$ (where Σ_i^+ is a push-off of Σ_i formed using the unique 'unlinked' normal vector field on Σ_i).

This bears some relationship with notions considered in [**7**, **9**, **11**]. Namely, if the links L_0 and L_1 cobound in $S^3 \times [0,1]$ an embedded symmetric grope of height $n+1$ then they are (n)-$\mathbb{Z}_p$-cobordant for any p. Therefore the following proposition might yield additional obstructions to the links being concordant to a trivial link. In fact this influence can already be seen in [**3**].

Proposition 5.2. *If L_0 and L_1 are (n)-$\mathbb{Z}_p$-cobordant and*

$$A = \pi_1(S^3 - L_0), \quad \overline{A} = \pi_1(S^3 - L_1),$$

then

$$A/A^{(n)} \cong \overline{A}/\overline{A}^{(n)}.$$

Proof. We use the notation of Definition 5.1 and Proposition 5.2. Let $B = \pi_1(E)$. By hypothesis, for each $1 \leqslant i \leqslant m$ there exist symplectic bases of circles $\{a_{ij}, b_{ij}\}$ for Σ_i with their push-offs $\{a_{ij}^+, b_{ij}^+\}$ into E lying in $\pi_1(E)^{(n)} = B^{(n)}$. The key observation is that $H_2(E;\mathbb{Z}_p)$ is generated by the tori $\{a_{ij}^+ \times S_i^1, b_{ij}^+ \times S_i^1\}$, where S_i^1 is a fiber of the normal circle bundle to Σ_i, together with the m tori $L_0 \times S_i^1$ that live in $S^3 - L_0$. Thus the cokernel of the map $H_2(A;\mathbb{Z}_p) \to H_2(B;\mathbb{Z}_p)$ is generated by the former collections. Since $[a_{ij}^+] \in B^{(n)}$, a_{ij}^+ bounds a $B^{(n-1)}$-surface S_{ij} mapped into E (that is, $\pi_1(S_{ij}) \subset B^{(n-1)}$). If we cut open the torus $a_{ij}^+ \times S_i^1$ along a_{ij}^+ and adjoin two oppositely oriented copies of S_{ij}, then we obtain a (mapped in) surface that is homologous to $a_{ij}^+ \times S_i^1$ and is also a $B^{(n-1)}$-surfaces, and similarly for the tori $b_{ij}^+ \times S_i^1$. Therefore $A \to B$ satisfies the hypotheses of Theorem 4.2 for $n-1$. Symmetrically, the same is true for $\overline{A} \to B$. The theorem follows immediately. □

References

1. M. Boileau and S. Wang, 'Non-zero degree maps and surface bundles over S^1', *J. Differential Geom.* 43 (1996) 789–806.
2. A. Bousfield, *Homological localization towers for groups and π-modules*, Memoirs of the American Mathematical Society 186 (American Mathematical Society, Providence, RI, 1977).
3. J. C. Cha, 'Link concordance, homology cobordism, and Hirzebruch-type intersection form defects from towers of iterated p-covers', Preprint, 2007, arXiv:math/0705.0088, *J. Eur. Math. Soc.* (to appear).
4. T. Cochran and S. Harvey, 'Homology and derived series of groups', *Geom. Topol.* 9 (2005) 2159–2191.
5. T. Cochran and S. Harvey, 'Homology and derived series of groups II: Dwyer's theorem', *Geom. Topol.* 12 (2008) 199–232.

6. T. Cochran and K. Orr, 'Homology cobordism and generalizations of Milnor's invariants', *J. Knot Theory Ramifications* 8 (1999) 429–436.
7. T. Cochran, K. Orr and P. Teichner, 'Knot concordance, Whitney towers and L^2-signatures', *Ann. of Math.* 157 (2003) 433–519.
8. T. Cochran, K. Orr and P. Teichner, 'Structure in the classical knot concordance group', *Comment. Math. Helv.* 79 (2004) 105–123.
9. T. Cochran and P. Teichner, 'Knot concordance and von Neumann ρ-invariants', *Duke Math. J.* 137 (2007) 337–379.
10. J. Conant and P. Teichner, 'Grope cobordism and Feynman diagrams', *Math. Ann.* 328 (2004) 135–171.
11. J. Conant and P. Teichner, 'Grope cobordism of classical knots', *Topology* 43 (2004) 119–156.
12. Marc Culler and Peter B. Shalen, 'Singular surfaces, mod 2 homology, and hyperbolic volume, ii', Preprint, 2007, arXiv/math.GT/0701666.
13. W. Dwyer, 'Homology, Massey products and maps between groups', *J. Pure Appl. Algebra* 6 (1975) 177–190.
14. M. H. Freedman and P. Teichner, '4-manifold topology II: Dwyer's filtration and surgery kernels', *Invent. Math.* 122 (1995) 531–557.
15. M. H. Freedman and F. Quinn, *The topology of* 4*-manifolds*, Princeton Mathematical Series 39 (Princeton University Press, Princeton, N.J., 1990).
16. J. Howie, 'Free subgroups in groups of small deficiency', *J. Group Theory* 1 (1998) 95–112.
17. S. Harvey, 'Homology cobordism invariants of 3-manifolds and the Cochran–Orr–Teichner filtration of the link concordance group', *Geom. Topol.* 12 (2008) 387–430.
18. P. J. Hilton and U. Stammbach, *A course in homological algebra*, Graduate Texts in Mathematics 4 (Springer, New York, 1971).
19. J. Howie and H. R. Schneebeli, 'Homological and topological properties of locally indicable groups', *Manuscripta Math.* 44 (1983) 71–93.
20. A. Kawauchi, 'Almost identical imitations of $(3,1)$-dimensional manifold pairs', *Osaka J. Math.* 26 (1989) 743–758.
21. A. Kawauchi, 'An imitation theory of manifolds', *Osaka J. Math.* 26 (1989) 447–464.
22. V. S. Krushkal, 'On the relative slice problem and and four-dimensional topological surgery', *Math. Ann.* 315 (1999) 363–396.
23. V. S. Krushkal, 'Dwyer's filtration and the topology of four-manifolds', *Math. Res. Lett.* 10 (2003) 247–251.
24. V. S. Krushkal and P. Teichner, 'Alexander duality, gropes and link homotopy', *Geom. Topol.* 1 (1997) 51–69.
25. M. Lackenby, 'Large groups, property (tau) and the homology growth of subgroups', Preprint, 2005 arXiv:math.GR/0509036.
26. M. Lackenby, 'New lower bounds on subgroup growth and homology growth', Preprint, 2005, arXiv:math.GR/0512261, *Proc. Lond. Math. Soc.* (to appear).
27. M. Lackenby, 'Detecting large groups', Preprint, 2007, arXiv:math.GR/0702571.
28. A. Lubotzky and D. Segal, *Subgroup growth*, Progress in Mathematics 212 (Birkhäuser, Basel, 2003).
29. D. Ruberman, 'Seifert surfaces of knots in S^4', *Pacific J. Math.* 145 (1990) 97–116.
30. P. B. Shalen and P. Wagreich, 'Growth rates, Z_p-homology, and volumes of hyperbolic 3-manifolds', *Trans. Amer. Math. Soc.* 331 (1992) 895–917.
31. J. Stallings, 'Homology and central series of groups', *J. Algebra* 2 (1965) 170–181.
32. J. Stallings, 'Surfaces in three-manifolds and non-singular equations in groups', *Math. Zeit.* 184 (1983) 1–17.
33. R. Strebel, 'Homological methods applied to the derived series of groups', *Comment. Math.* 49 (1974) 302–332.
34. P. Teichner, 'Knots, von Neumann signatures and grope cobordism', *Proceedings of the ICM*, Beijing, Vol. II (Higher Education Press, 2002) 437–446 (Invited Lectures).

Tim Cochran and Shelly Harvey
Mathematics Department
Rice University
PO Box 1892
MS-136, Houston, TX 77005-1892
USA

cochran@rice.edu
shelly@rice.edu