On Transverse Knots and Branched Covers

Shelly Harvey[1], Keiko Kawamuro[1], and Olga Plamenevskaya[2]

[1]Department of Mathematics, Rice University, Houston, TX 77005, USA, and [2]Deparment of Mathematics, Stony Brook University, Stony Brook, NY 11790, USA

Correspondence to be sent to: olga@math.sunysb.edu

We study contact manifolds that arise as cyclic branched covers of transverse knots in the standard contact 3-sphere. We discuss properties of these contact manifolds and describe them in terms of open books and contact surgeries. In many cases we show that such branched covers are contactomorphic for smoothly isotopic transverse knots with the same self-linking number. These pairs of knots include most of the nontransversely simple knots of Birman–Menasco and Ng–Ozsváth–Thurston.

1   Introduction

In this paper, we consider transverse knots in (S^3, ξ_{std}), i.e. knots that are transverse to the contact planes of the standard contact structure $\xi_{std} = \ker(dz - y\,dx)$.

A simple "classical" invariant is given by the self-linking number *sl* of a transverse knot. However, if L_1, L_2 are two transverse knots that are smoothly isotopic and share the same self-linking number, L_1 and L_2 do not have to be transversally isotopic: this phenomenon was first discovered in [11] and [4], and more examples were recently obtained in [19].

The goal of this paper is to study contact manifolds that arise as cyclic covers branched over transverse knots and links in (S^3, ξ_{std}). Such cyclic covers carry

natural contact structures lifting ξ_{std}. The main question we would like to address is:

Question 1.1. Suppose that transverse knots L_1, L_2 are smoothly isotopic, and $sl(L_1) = sl(L_2)$. Fix $p \geq 2$. Are p-fold cyclic covers branched over L_1 and L_2 contactomorphic? □

Finding two such non-contactomorphic covers would imply that the induced contact structure on the branched cover is an effective invariant of transverse knots. On the other hand, a positive answer to the above question for any pair of knots means that the cyclic branched covers are insensitive to the subtler structure of transverse knots.

While we found no examples of non-contactomorphic branched covers, we are able to answer Question 1.1 positively in many special cases. In particular, we show that branched cyclic covers of any degree are contactomorphic for all examples of Birman–Menasco [4, 5], and that branched double covers for many examples of [19] are also contactomorphic. We prove:

Theorem 1.2. The p-fold cyclic branched covers of transverse links L_1 and L_2 are contactomorphic for all p if:

- $L_1 = L^+$ and $L_2 = \bar{L}^-$ are a positive and a negative transverse push-offs of a Legendrian link L and its Legendrian mirror $\bar{L}$, or
- L_1 and L_2 are given by transverse 3-braids related by a negative flype.

Moreover, the branched double covers are contactomorphic for arbitrary transverse braids related by a negative flype. □

In fact, we will prove a little more in Section 5. We also note that all examples of Birman–Menasco satisfy the second condition of Theorem 1.2, where a *negative flype* (Figure 24) is a braid move introduced in [4].

Let $\xi_p(L)$ denote the natural contact structure on the branched p-fold cyclic cover $\Sigma_p(L)$ of a transverse link L as explained in Section 2.5. We describe the contact manifolds $\Sigma_p(L)$ in two ways.

First, we give an open-book decomposition supporting $\xi_p(L)$. If L is represented as a transverse n-braid, an open-book decomposition of $(\Sigma_p(L), \xi_p(L))$ can be obtained as a lift of the open book for S^3 whose binding is the braid axis, and a page is a disk meeting L transversely at n points. The monodromy for the resulting open book can be read off the braid word. More precisely, a positive crossing in the braid contributes $(p-1)$ positive

Dehn twists to the monodromy, while a negative crossing contributes $(p-1)$ negative Dehn twists.

Second, we give contact surgery diagrams [6, 7] for these contact manifolds. We find that a positive (resp. negative) crossing in the braid corresponds to a Legendrian surgery (resp. $(+1)$ contact surgery) on $(p-1)$ standard Legendrian unknots. Interestingly, it turns out that the linking between these $(p-1)$ unknots depends on the sign of the crossing: while for a negative crossing the surgery is performed on *unlinked* unknots (Figure 13), for a positive crossing the unknots are linked (Figure 12). This phenomenon arises in the smooth setting as well, where the construction can be thought of as a version of the Montesinos trick for higher order covers. We refer the reader to Lemma 3.1 and Theorem 3.4 for precise statements.

This description yields a few properties of p-fold cyclic branched covers; we can determine whether they are tight or overtwisted in certain cases and describe the homotopy invariants of the contact structures.

Theorem 1.3.   The contact manifold $(\Sigma_p(L), \xi_p(L))$ is Stein fillable if the transverse link L is represented by a quasipositive braid; it is overtwisted if L is obtained as a transverse stabilization of another transverse link.   □

In fact, in Section 4 we prove overtwistedness for a much wider class of contact structures.

Theorem 1.4.   Fix $p \geq 2$. Let $\mathfrak{s}_L$ be the Spin^c structure induced by $\xi = \xi_p(L)$. Then $c_1(\mathfrak{s}_L) = 0$. The three-dimensional invariant $d_3(\xi)$ is completely determined by the topological link type of L and its self-linking number $sl(L)$.   □

The present paper continues the work started by the third author in [22], where Question 1.1 was studied for the case of branched double covers. The paper [22] was written before the advent of Heegaard Floer transverse invariants [20], and the only explicit examples of non-transversely simple knots available then were the 3-braids of [4]. The techniques from [22] are useful for the higher order covers as well; in particular, Theorems 1.4 and 1.3 are direct extensions of results of [22].

2   Preliminaries

In this section, we fix notation and collect the necessary facts about transverse knots, open-book decompositions, and contact surgeries, referring the reader to [7, 9, 10] for

details. We assume that all 3-manifolds are closed, connected and oriented, and all contact structures are co-oriented.

2.1   Transverse knots as braids

It will be helpful to represent transverse links by closed braids. For this, consider the symmetric version of the standard contact structure (S^3, ξ_{sym}) with $\xi_{sym} = \ker(dz + x\,dy - y\,dx)$. Then, any closed braid about z-axis can be made transverse to the contact planes; moreover, any transverse link is transversely isotopic to a closed braid in (S^3, ξ_{sym}) [2]. Equivalently, we can consider transverse braids in the contact structure $\xi_{std} = \ker(dz - y\,dx)$; for example, assuming that our braids are satellites of a fixed standard Legendrian unknot with $tb = -1$.

To define the self-linking number $sl(L)$, trivialize the plane field ξ, and let the link L' be the push-off of L in the direction of the first coordinate vector for ξ. Then, $sl(L)$ is the linking number between L and L'. Given a closed braid representation of L, we have

$$sl(L) = n_+ - n_- - b, \tag{2.1}$$

where n_+ (n_-) is the number of positive (negative) crossings, and b is the braid index.

The *stabilization* of a transverse link represented as a braid is equivalent to the negative braid stabilization, i.e. adding an extra strand and a negative kink to the braid. If L_{stab} is the result of stabilization of L, then

$$sl(L_{stab}) = sl(L) - 2. \tag{2.2}$$

Note that the positive braid stabilization does not change the transverse type of the link.

Abusing notation, we will often identify a transverse link with its braid word, writing it in terms of the standard generators $\sigma_1, \sigma_2, \ldots$ and their inverses.

Another useful way to think about transverse knots is as push-offs of Legendrian knots. Indeed, a given Legendrian knot yields two transverse knots, a positive push-off and a negative push-off, whose self-linking number is $tb(L) \pm r(L)$, respectively. This description is used in [19, 20].

2.2   Open books

An open-book decomposition of a 3-manifold M is a pair (S, ϕ) of a surface S with nonempty boundary ∂S and a diffeomorphism ϕ of S with $\phi|_{\partial S} = id$, such that $M \setminus \partial S$ is
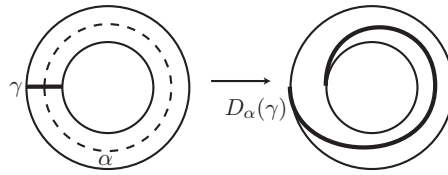
Fig. 1.   A right-handed Dehn twist D_α about α.

the mapping torus $S \times [0, 1]/ \sim$, where $(x, 1) \sim (\phi(x), 0)$. The surface S is called a *page* and ∂S the *binding* of the open book. By the celebrated work of Giroux [13], contact structures on M up to an isotopy are in one-to-one correspondence with open-book decompositions of M up to positive stabilization. A positive stabilization of an open book consists of plumbing a right-handed Hopf band, i.e. attaching a 1-handle to a page and composing the monodromy with a right-handed Dehn twist along an arbitrary curve intersecting the cocore of the handle at one point. A right-handed Dehn twist D_α about a simple closed curve $\alpha \subset S$ is a diffeomorphism that acts on a neighborhood $N = \alpha \times (0, 1) \subset S$ of α as $(\theta, t) \mapsto (\theta + 2\pi t, t)$, fixing $S \setminus N$; see Figure 1. The term "positive Dehn twist" is also common in the literature, but we avoid it since positive Dehn twists will correspond to (-1) contact surgeries. A left-handed Dehn twist about α is the inverse of D_α.

We recall that the monodromy of an arbitrary open book can be written as a product of left- and right-handed Dehn twists, and that a contact structure is Stein fillable if and only if it admits an open book with the monodromy given by a product of right-handed Dehn twists [13].

2.3   Contact surgery

Let K be a null-homologous Legendrian knot in a contact manifold (Y, ξ). Performing a Dehn surgery on K, we cut out a tubular neighborhood of the knot K (i.e. a solid torus) and glue it back in. When the surgery coefficient is (± 1) with respect to the contact framing on K, this procedure is compatible with contact structures: the gluing can be done so that the contact structure on the solid torus matches the contact structure on its complement. Moreover, the resulting contact manifold is independent of choices, so the (± 1) contact surgery is well defined. Contact surgery is a very useful tool, as any contact manifold can be obtained from (S^3, ξ_{std}) by a contact surgery on some Legendrian link. We also recall that (-1) contact surgery is in fact the same as Legendrian surgery, while $(+1)$ contact surgery is the operation inverse to it. Unlike Legendrian surgery, $(+1)$ surgery does not preserve Stein fillability or other similar properties of contact structures.

Homotopy invariants of a contact structure ξ on Y encode information on the corresponding plane field. First, we can consider the Spin^c structure $\mathfrak{s}$ on Y induced by ξ. Second, when $c_1(\mathfrak{s})$ is torsion, the three-dimensional invariant $d_3(\xi)$ can be defined [14]. If (Y, ξ) is the boundary of an almost-complex 4-manifold (X, J), this invariant is given by

$$d_3(\xi) = \tfrac{1}{4}\big(c_1^2(J) - 2\chi(X) - 3\operatorname{sign}(X)\big).$$

The homotopy invariants of a contact structure can be read off its contact surgery presentation as follows [7]. Let X be the four-manifold obtained from D^4 by attaching the 2-handles as dictated by the (± 1)-surgery diagram. Consider an almost-complex structure J defined on X in the complement of m balls lying in the interior of the $(+1)$-surged 2-handles of X. As shown in [7], J induces a Spin^c structure $\mathfrak{s}_J$, which extends to all of X, and the d_3 invariant of ξ can be computed as

$$d_3(\xi) = \tfrac{1}{4}\big(c_1^2(\mathfrak{s}_J) - 2\chi(X) - 3\operatorname{sign}(X)\big) + m. \tag{2.3}$$

This formula is very similar to the case where (X, J) is almost-complex, except that there is a correction term of $(+1)$ for each $(+1)$-surgery.

Now, suppose that a 2-handle is attached to the four-manifold X in the process of Legendrian surgery on a knot K, and denote by $[S]$ the homology class that arises from the Seifert surface of K capped off inside the handle. It is well known [14] that $c_1(\mathfrak{s}_J)$ evaluates on $[S]$ as the rotation number of the Legendrian knot K. Furthermore, it is shown in [7] that the same result is true for $(+1)$-contact surgeries (for the Spin^c structure $\mathfrak{s}_J$ on X described above).

2.4   Surgery descriptions from open books

There are two ways to describe a given contact structure, via an open-book decomposition or a contact surgery diagram. We will need to switch between the two descriptions. A contact surgery diagram consists of a Legendrian link in S^3 with surgery coefficients. We can find an open-book decomposition of S^3 whose page contains this link. Thus components of the surgery link correspond to curves on the page; we perform right-handed Dehn twists on curves corresponding to Legendrian surgeries, and left-handed Dehn twists on those corresponding to $(+1)$ contact surgeries. The resulting open book is compatible with the contact structure given by the surgery diagram [1, 10, 21]. Conversely, given an open-book decomposition of a given contact manifold, we will need to obtain a

contact surgery diagram. To this end, we assume that the monodromy of the open book contains a sequence of Dehn twists producing (S^3, ξ_{std}) (this can always be achieved by composing the given monodromy with a few Dehn twists and their inverses). We can then embed the page of the open book into S^3; if the curves on which the remaining Dehn twists are to be performed become Legendrian knots in S^3, we can replace the Dehn twists with (± 1) contact surgeries to obtain the required surgery diagram. (Note that a "compatible" embedding will imply that the contact framing of the Legendrian knots is the same as the page framing.) We perform this procedure in detail in Section 3.2.

2.5   The induced contact structure on $\Sigma_p(K)$

Given a p-fold cyclic branched cover $\Sigma_p(K)$ for a transverse knot K, we describe the natural contact structure $\xi_p(K)$ on $\Sigma_p(K)$ as follows. In local coordinates (r, θ, z) near the knot $K = \{r = 0\}$ we can assume that it has a contact structure $\xi = \ker(dz + r^2\, d\theta)$. We write the covering map as $(r, \theta, z) \mapsto (r^p, p\theta, z)$. Set $\xi_p(K)$ to be the kernel of the pull-back form; however, the pull-back form $dz + pr^{2p}\, d\theta$ fails to be contact along the knot. To avoid this issue, we can define a new contact form by interpolating between the form $dz + r^2 d\theta$ in a small tubular neighborhood of K and the pull-back form on the branched cover away from K. Its kernel is a contact structure, which is independent of choices. (This construction is explained in detail in [22] for branched double covers and works for links and higher order covers with only notational changes.)

We can also describe the contact structure on $\Sigma_p(K)$ via open books, by representing K as a braid. We then consider a branched cover of the standard open book for S^3 whose binding is the braid axis, and page a disk meeting the n-braid K at n points. We adopt this approach in the next section, determining how the half-twist generators of the braid K lift to the branched cover. It is clear that the resulting contact structure is isotopic to the one described above.

3   Open Books and Surgeries from Braids

3.1   Dehn twists and crossings

Let $K \subset (S^3, \xi_{sym})$ be a transverse link. Identifying K with a closed braid of braid index n about the z-axis, let $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_K} \in B_n$ denote a braid representation of K. Let $D = \{(r, \theta, z) | \theta = 0, r > 0\} \subset \mathbb{R} \cup \{\infty\} = S^3$ be a disk. Then K transversely intersects D in n
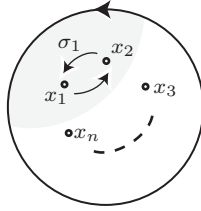
Fig. 2. A page D and the map σ_1 acting on the shaded region U_1.

points $x_1, \ldots, x_n$. We may regard $\sigma_j \in B_n$ as a diffeomorphism of D that exchanges x_j, x_{j+1}, as in Figure 2, in the neighborhood U_j of x_j, x_{j+1} and fixes $D \setminus U_j$.

Let $\phi_K = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_K}$ be a monodromy map of D. The symmetric contact structure (S^3, ξ_{sym}) is supported by the open-book decomposition (D, ϕ_K) of S^3, whose binding is the z-axis (braid axis) and pages are disks D.

Fix $p \geq 2$, and let $\pi : \Sigma_p(K) \to S^3$ be the p-fold cyclic covering branched along K. The covering π induces the open-book decomposition $(\tilde{D}, \tilde{\phi}_K) = (\pi^{-1}(D), \pi^{-1}(\phi_K))$ of $\Sigma_p(K)$ given by the lift of the open book (D, ϕ_K). The surface $\tilde{D}$ can be obtained by gluing p copies of D along slits as in Figure 3. There, the labels $a_{j,k}$ on the boundary of the cut-up disks help to specify how the copies of D are glued together. For example, denoting $\tilde{x}_j = \pi^{-1}(x_j) \in \tilde{D}$, we identify the edge $a_{j,k}\tilde{x}_j$ of the kth sheet with the edge $a_{j,k}\tilde{x}_j$ of the $(k+1)$th sheet.

To compute the monodromy $\tilde{\phi}_K$, we need the following lemma that describes the lift of σ_1. This lift will be given by a composition of Dehn twists along curves α_k shown in Figure 4. The curve α_k lies in the union of the kth and $(k+1)$th sheets.

Lemma 3.1. Let $\alpha_k \subset \tilde{D}$ where $k = 1, \ldots, p-1$ be a simple closed curve as in Figure 4. Let $D_k = D_{\alpha_k}$ be the right-handed Dehn twist about α_k. Then the lift $\tilde{\sigma}_1$ of σ_1 is $D_1 \circ D_2 \circ \cdots \circ D_{p-1}$ (where D_{p-1} comes first and D_1 last). $\qquad\square$

Proof. For simplicity, denote $\sigma := \sigma_1$ and $U := U_1$. We need to show that up to isotopy,

$$\pi^{-1} \circ \sigma^{-1} \circ \pi \circ D_1 \circ D_2 \circ \cdots \circ D_{p-1} = id_{\tilde{D}}. \tag{3.1}$$

Cut $\tilde{D}$ into $(n + p)$ disks along oriented properly embedded arcs $\lambda_j^{(i)}$ where $i = 1, \ldots, p$ and $j = 1, \ldots, n-1$, dashed in Figure 4. We will check that up to an isotopy, the map $\pi^{-1} \circ \sigma^{-1} \circ \pi \circ D_1 \circ D_2 \circ \cdots \circ D_{p-1}$ fixes each vertex and edge of the graph $\cup_{i,j} \lambda_j^{(i)}$. Our statement will then follow from the Alexander method [12, Proposition 3.4], which is
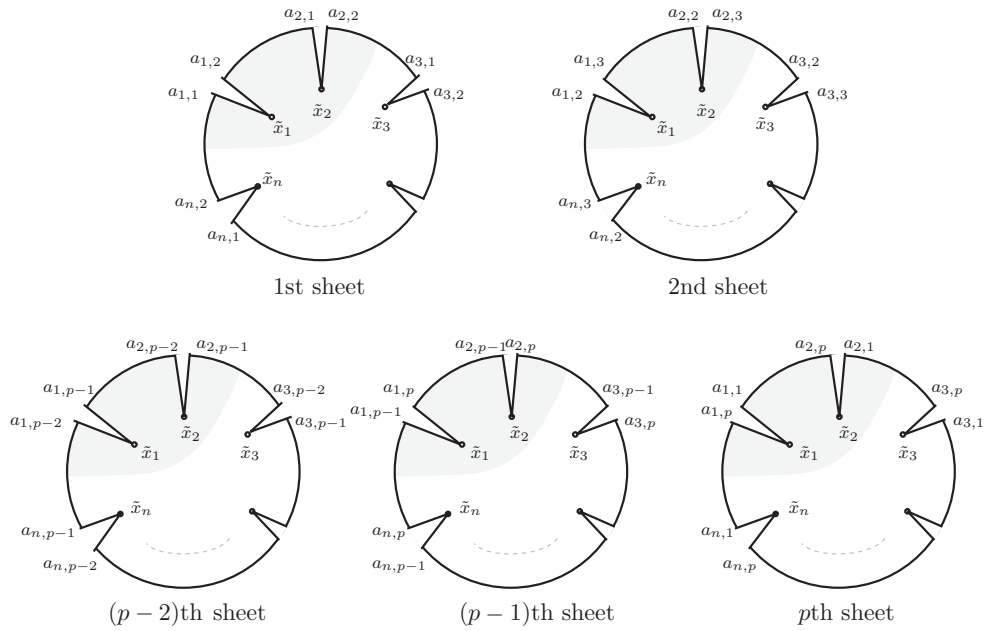
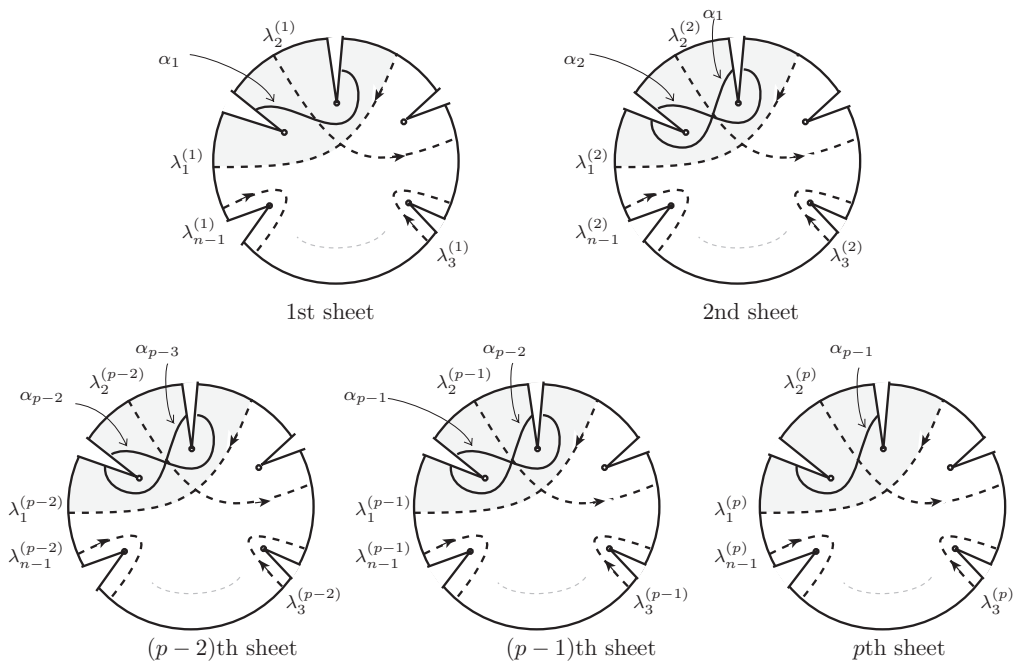**Fig. 3.** Construction of a page $\tilde{D}$. The region $\pi^{-1}(U_1)$ is shaded.

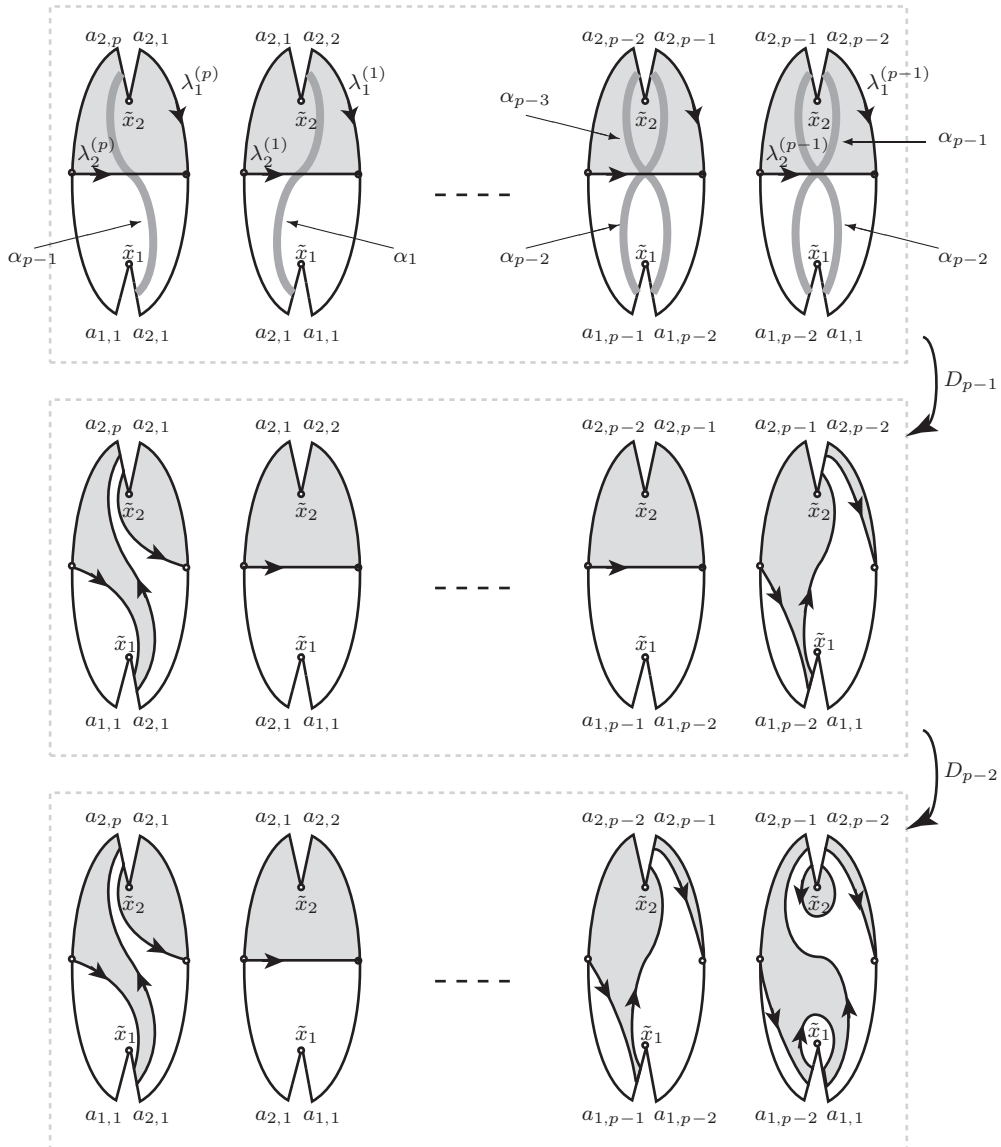**Fig. 4.** A simple closed curve α_k and an arc $\lambda_j^{(i)}$.

Fig. 5. Actions of D_{p-1} and D_{p-2} on $\pi^{-1}(U)$.

based on the observation that a diffeomorphism of D fixing ∂D is isotopic to the identity. This observation is applied to each of the $(n + p)$ disks.

Since the Dehn twists are performed on curves $\alpha_1, \ldots, \alpha_{p-1}$, which all lie in $\pi^{-1}(U)$, we can assume that all the λ-arcs except $\lambda_2^{(i)}$'s are fixed by $D_1 \circ D_2 \circ \cdots \circ D_{p-1}$. Therefore, we focus on $\pi^{-1}(U)$ shown in the top box of Figure 5 to understand how the arcs $\lambda_2^{(i)}$ change under the map $\pi^{-1} \circ \sigma^{-1} \circ \pi \circ D_1 \circ D_2 \circ \cdots \circ D_{p-1}$. Note that in Figure 5, the region $\pi^{-1}(U)$
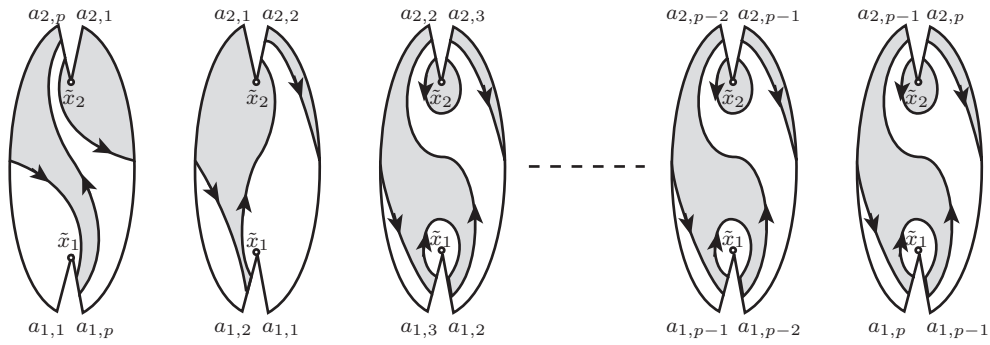
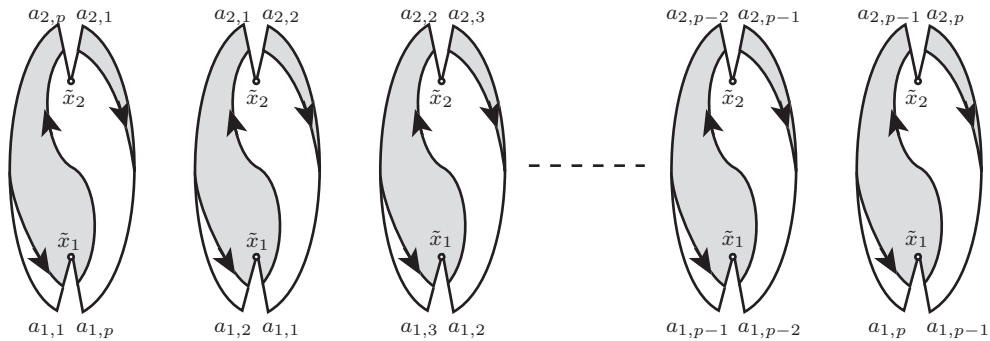**Fig. 6.** The region $W = D_1 \circ D_2 \circ \cdots \circ D_{p-1}(\pi^{-1}(U))$.

Fig. 7. The region W' obtained by finger moves applied to W.

is separated into two regions by the λ arcs and the left side of λ's is shaded. The Dehn twist D_{p-1} changes the region $\pi^{-1}(U)$, shown in the top box of Figure 5, to $D_{p-1}(\pi^{-1}(U))$ as in the second box. Then D_{p-2} changes it to $D_{p-2}(D_{p-1}(\pi^{-1}(U)))$ as in the bottom box. Applying all the Dehn twists $D_1 \circ D_2 \circ \cdots \circ D_{p-1}$ to $\pi^{-1}(U)$, we obtain the region W shown in Figure 6. Next we isotope W fixing the boundary of W by two local finger moves near $\tilde{x}_1$ and $\tilde{x}_2$, and obtain a region W' as in Figure 7.

To complete the proof, we observe that the region W' is precisely $\pi^{-1}(\sigma(U))$. ∎

Applying Lemma 3.1 repeatedly for different pairs of points x_j, x_{j+1}, we can write down the monodromy of an arbitrary braid. We denote the curve α_k introduced in Lemma 3.1 by α_k^j ($k = 1, \ldots, p-1$, $j = 1, \ldots, n-1$) when it is related to the twist of branch points x_j, x_{j+1} and lies on the kth and $(k+1)$th sheets, and write D_k^j for the
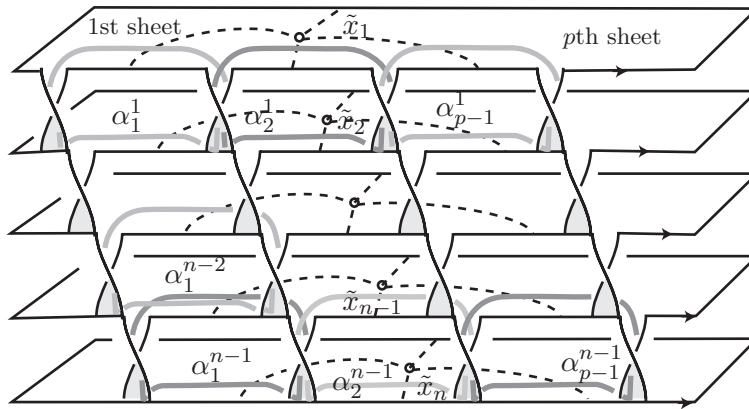
Fig. 8. A page $\tilde{D}$ of the open book with simple closed curves α_k^j where $p = 4$ and $n = 5$.

right-handed Dehn twist about α_k^j. In particular, the α_k curve in Figure 4 is renamed as α_k^1, and the corresponding Dehn twist is D_k^1.

Proposition 3.2. Let K be the braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$. Then the open book of the p-fold cover branched over K given by Lemma 3.1 is the same as the open book of S^3 induced by the (n, p)-torus link fibration; moreover, the images of the curves α_k^j on the Seifert surface of this torus link are as shown on Figure 8. Each α_k^j is an unknot in S^3, with page framing $= -1$. □

Proof. We first observe that $(\Sigma_p(K), \xi_p(K)) = (S^3, \xi_{std})$. This is easy to see: since K is the transverse unknot with $sl = -1$, it can be thought of as the binding of an open-book decomposition of S^3 whose page is a disk. The branched p-fold cover, then, is given by the same open book for any p, yielding the standard contact structure on S^3.

Lemma 3.1 produces a different open book for $\Sigma_p(K)$. A page of this open book, together with the curves α_k^j, can be embedded into S^3 as a Seifert surface of the (p, n)-torus link shown on Figure 8. It is then clear that each α_k^j is an unknot, with page framing $=-1$. We claim that the torus knot fibration induces the monodromy of the open book given by Lemma 3.1, i.e. the monodromy of this torus knot is the product of the Dehn twists $(D_1^{n-1} \circ \cdots \circ D_{p-1}^{n-1}) \circ \cdots \circ (D_1^2 \circ \cdots \circ D_{p-1}^2) \circ (D_1^1 \circ \cdots \circ D_{p-1}^1)$. Since the fiber surface of the torus knot can be obtained by plumbing together a sequence of right-handed Hopf bands whose core curves are α_k^j, it is clear that the monodromy of the torus knot is given by a composition of the right-handed Dehn twists D_k^j. We need to determine the order in which the Dehn twists are performed.
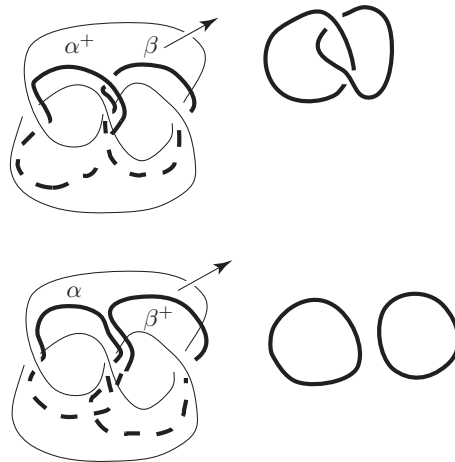
Fig. 9.   The curves α and β lie on a fiber of the trefoil knot fibration; α^+ and β^+ are their push-offs in the positive normal direction. (The positive normal points out of the page toward the reader.)

To simplify the picture, we consider a model example where $n = 2$, $p = 3$. See Figure 9. Let T the right-handed trefoil and consider the fibration $S^3 \setminus T \to S^1$. Its monodromy is the product of the Dehn twists around the curves $\alpha = \alpha_1^1$ and $\beta = \alpha_2^1$. Let P_θ, $\theta \in [0, 2\pi)$ be pages of the corresponding open book. Assume that the curves α and β both lie on P_0; let α^+ and β^+ be their push-offs to the page P_{θ^+} for some small $\theta^+ > 0$. Since $S^3 \setminus T$ is oriented as a mapping torus, this means that the curves are pushed off in the direction shown by arrow in Figure 9. Observe that α^+ and β form a Hopf link, while α and β^+ are not linked. Suppose that the monodromy of the pictured trefoil is $D_\beta \circ D_\alpha$, and compose it with $D_\alpha^{-1} \circ D_\beta^{-1}$. The result is of course the open book with trivial monodromy, which gives $\#_2 S^1 \times S^2$. On the other hand, the composition of the two additional Dehn twists corresponds to an integral surgery on S^3 performed on the link $\alpha^+ \cup \beta$ (since D_α^{-1} follows D_β^{-1}, we need to place a copy of α on the page following the page with β). The surgery coefficients are given by (page framing)$+1$, so we perform 0-surgery on both α^+ and β; but this surgery on the Hopf link produces S^3, not $\#_2 S^1 \times S^2$. By contrast, if we perform 0-surgeries on α and β^+, which form a trivial link (and correspond to composing the trefoil monodromy with $D_\beta^{-1} \circ D_\alpha^{-1}$), we obtain $\#_2 S^1 \times S^2$, so we conclude that the trefoil monodromy is $D_\alpha \circ D_\beta$.

A similar argument for various pairs of curves α_k^j shows that the monodromy of the torus knot on Figure 8 is indeed $(D_1^{n-1} \circ \cdots \circ D_{p-1}^{n-1}) \circ \cdots \circ (D_1^2 \circ \cdots \circ D_{p-1}^2) \circ (D_1^1 \circ \cdots \circ$

D_{p-1}^1). The curves α_k^j and $\alpha_l^{i\,+}$, the push-off of α_l^i, form a Hopf link whenever

$$(i, l) = (j, k), (j + 1, k - 1), (j, k - 1), \text{ or } (j + 1, k),$$

and the unlink otherwise. See Figure 11.   ∎

3.2   Surgery diagrams for branched covers

Open books from the previous section will allow us to construct contact surgery diagrams for the branched covers. In Proposition 3.2, we saw that the branched p-fold cover for the transverse braid $K = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$ is (S^3, ξ_{std}). Now consider a transverse n-braid $L = \sigma_1 \sigma_2 \cdots \sigma_{n-1} b$, where b is an arbitrary braid word. The branched p-fold cover for L can be obtained from the branched cover for K by performing additional Dehn twists about curves α_j^i considered in Lemma 3.1.

The goal of this subsection is to interpret these Dehn twists as contact surgeries. Forgetting the contact structure, we can translate Dehn twists into Dehn surgeries along push-offs of the curves α_j^i to successive pages of our open book. A left-handed (resp. right-handed) Dehn twist gives a surgery with coefficient (page framing) $+ 1$ (resp. (page framing) $- 1$). By Proposition 3.2 the page framing of each α_j^i is -1, so we perform 0-surgeries for left-handed and (-2)-surgeries for right-handed Dehn twists. The order of push-offs is determined by the order of Dehn twists, which in turn is dictated by the braid word b and Lemma 3.1.

Using Honda's Legendrian realization [16], we can in principle find an isotopy that takes all α_j^i to Legendrian curves whose contact framing matches the page framing, so that 0- and (-2)-surgeries become contact (± 1)-surgeries. This is almost what we need, but we want an explicit surgery diagram; to this end, we give an explicit Legendrian realization of our curves. Indeed, following [1] (see [21, Appendix] for the same construction in the presence of a contact structure), we can embed the fiber surface of a torus link (Figure 8) into S^3 as the page $P_0 (= P_{\theta=0})$ of an open-book decomposition compatible with ξ_{std}, and such that α_j^i are all Legendrian unknots with $tb = -1$. We simply draw this surface as in Figure 10, assuming as usual that $\xi_{std} = \ker(dz - y\,dx)$. Various Legendrian push-offs of α_j^i can then be thought of as lying on different pages of the same open book.

To produce a contact surgery diagram of the p-fold branched cover for a transverse braid $L = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) b$, we start with S^3, write down the monodromy of the open book as dictated by b and Lemma 3.1, and then perform Legendrian surgeries on the
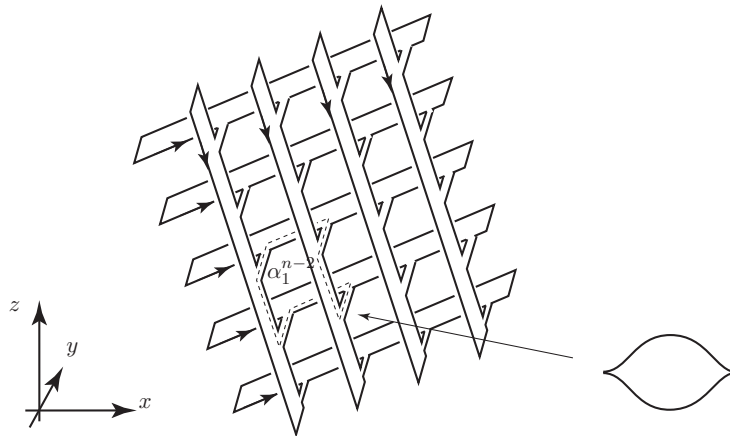
Fig. 10.   A Legendrian realization of Figure 8.

successive Legendrian push-offs of α_j^i's, in the order corresponding to the order of Dehn twists in the decomposition of the monodromy.

Remark 3.3.   In certain cases, it is easy to see that the push-offs of different curves α_j^i will be unlinked even if the curves themselves intersect on the surface P_0. Indeed, consider the braid $K = \sigma_1 \cdots \sigma_{n-1}$ and the braid $K' = \sigma_1 \cdots \sigma_{n-1} \sigma_j$, which differs from K by an additional crossing. The links K and K' differ only in a small ball B that contains this crossing; the p-fold branched cover of B is a genus $(p-1)$ handlebody, and the contact manifolds $\Sigma_p(K) = S^3$ and $\Sigma_p(K')$ differ only by a surgery on this handlebody. In fact, the surgery on the handlebody is equivalent to $(p-1)$ surgeries on the push-offs of α_k^j where $k = 1, \ldots, p-1$, corresponding to the crossing σ_j; the surgery curves are all contained in the handlebody. We also observe that if B is a neighborhood of an arc joining the two braid strands of K at the extra crossing, then the p-fold branched cover of B over the crossing is equal to a neighborhood of the p-fold branched cover of this joining arc.

Now, let c_1 and c_2 be two extra crossings added to K, and a_1 and a_2 the corresponding arcs. Untwisting the unknot K, we can easily determine whether the lifts of a_1 and a_2 to the branched cover are linked; if they are not, the corresponding surgery curves will not be linked either. If, however, a_1 and a_2 are linked, we have to examine the push-offs of the related curves α_k^j to determine the surgery link.   □

In view of Remark 3.3, it remains to examine a few cases where the push-offs of α_k^j's are linked. We orient α_k^j so that it goes from $\tilde{x}_j$ to $\tilde{x}_{j+1}$ on the kth sheet of Figure 4.
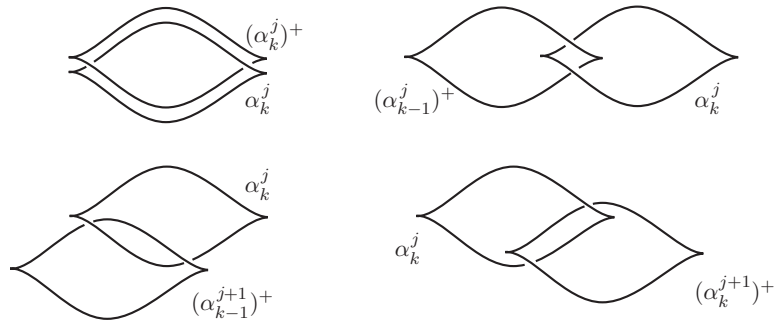
Fig. 11.   Legendrian push-offs of various curves α_k^j. In all cases not shown, α_k^j and $(\alpha_l^i)^+$ do not link.

We have

$$lk\big(\alpha_k^j, (\alpha_k^j)^+\big) = lk\big(\alpha_k^j, (\alpha_{k-1}^{j+1})^+\big) = -1,$$
$$lk\big(\alpha_k^j, (\alpha_{k-1}^j)^+\big) = lk\big(\alpha_k^j, (\alpha_k^{j+1})^+\big) = +1.$$

See Figure 11. In all other cases, the curves α_k^j and $(\alpha_l^i)^+$ do not link to each other.

Given a transverse n-braid L, we can always write it in the form $L = (\sigma_1\sigma_2\cdots\sigma_{n-1})b$ (possibly after multiplying by the trivial word $\sigma_1\cdots\sigma_{n-1}\sigma_{n-1}^{-1}\cdots\sigma_1^{-1}$). We start with an open book for S^3 corresponding to the braid $\sigma_1\cdots\sigma_{n-1}$, and let $P_\theta, \theta \in [0, 2\pi)$ denote its pages. Then we use the above algorithm to construct a contact surgery diagram for the p-fold cover $\Sigma_p(L)$. It will be convenient to use notation $\Omega_p(L)$ for the corresponding framed Legendrian link; when p is fixed, we often drop it from notation. (Although our notation does not include this, we will often need to remember the embedding of the link $\Omega_p(L)$ into the open book for S^3.)

Examining the addition of an individual σ_k or σ_k^{-1} to the braid word for L, we obtain the following theorem.

Theorem 3.4. Fix $p \geq 2$. Let $L = (\sigma_1\sigma_2\cdots\sigma_{n-1})b$ be a transverse n-braid. Assume that the surgery link $\Omega(L)$ is contained in $\bigcup_{0<\theta<\theta_0} P_\theta$ for some $\theta_0 < 2\pi$. Pick $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{p-1} < 2\pi$. Denote the copy of α_j^i in the page P_θ of the open book by $\alpha_j^{i,\theta}$.

(1) Suppose $L^+ = (\sigma_1\sigma_2\cdots\sigma_{n-1})b\sigma_k$, $1 \leq k \leq n-1$. Define the diagram u_k^+ as in Figure 12. Then the surgery diagram obtained as the union of framed links $\Omega(L)$ and u_k^+ describes the contact manifold $\Sigma_p(L^+)$.
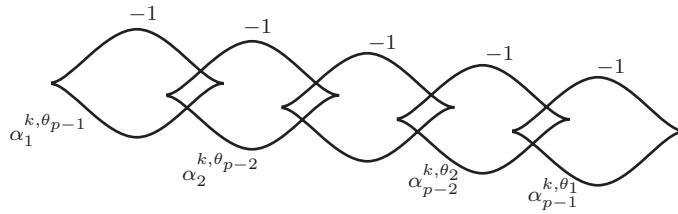
Fig. 12.   Contact surgery diagram u_k^+.

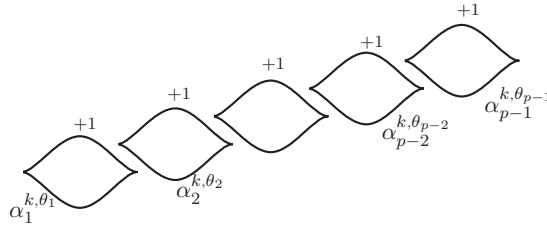**Fig. 13.**   Contact surgery diagram u_k^-.

(2) Suppose $L^- = (\sigma_1\sigma_2\cdots\sigma_{n-1})b\sigma_k^{-1}$, $1 \leq k \leq n-1$. Define the diagram u_k^- as in Figure 13. Then the surgery diagram obtained as the union of framed links $\Omega(L)$ and u_k^- describes the contact manifold $\Sigma_p(L^-)$.

(Here and below, we draw Legendrian links as their front projections to the (x, z) plane.)

The diagrams u_k^+ and u_k^- may link to $\Omega(L)$; the way they link can be determined by keeping track of the order of pages containing the link components and drawing the corresponding Legendrian push-offs of α_j^i as dictated by Figure 11.   □

Proof.   This is a direct application of the algorithm developed above.   ■

In addition to Theorem 3.4, we also apply our algorithm to a few other useful special cases.

Corollary 3.5.   Suppose that $L = \sigma_1\cdots\sigma_{n-1}b \in B_n$ and $L' = \sigma_1\cdots\sigma_{n-1}\sigma_n b \in B_{n+1}$ (i.e., L' is a positive braid stabilization of L representing the same transverse link). Then the branched covers of L and L' can be described by the same surgery diagram. Note that every positively stabilized braid can be written in such form.   □

Proof.   First, we consider two different initial open books for S^3: the one corresponding to the braid $\sigma_1\cdots\sigma_{n-1}$ and the other to $\sigma_1\cdots\sigma_{n-1}\sigma_n$. These yield empty surgery diagrams.
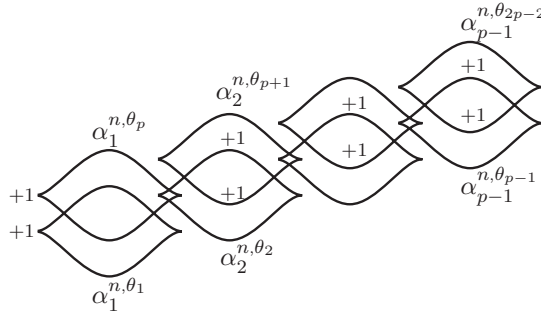
Fig. 14.   Contact surgery diagram u_n^{ot}.

Now, to build surgery diagrams for branched covers of the braids L and L', we have to add surgeries corresponding to generators in the word b. Step-by-step application of Theorem 3.4 ensures that the resulting diagrams will be the same. ∎

Corollary 3.6.   Let $L = \sigma_1 \cdots \sigma_{n-1} b \in B_n$, and $L_{stab} = \sigma_1 \cdots \sigma_{n-1} \sigma_n^{-1} b \in B_{n+1}$ (i.e., L_{stab} is a *negative* braid stabilization of L, representing a transverse link stabilization). Let u_n^{ot} be the contact surgery diagram shown on Figure 14. Then the branched cover $\Sigma_p(L_{stab})$ can be described by a surgery diagram, which is the union of $\Omega(L)$ and u_n^{ot}, where u_n^{ot} *does not* link $\Omega(L)$. The contact manifold represented by u_n^{ot} is an overtwisted 3-sphere. □

Proof.   We write $L_{stab} = \sigma_1 \ldots \sigma_{n-1} \sigma_n \sigma_n^{-2} b$. Applying part (2) of Theorem 3.4 twice, we see that the diagram u_n^{ot} describes the branched cover of the braid $\sigma_1 \cdots \sigma_{n-1} \sigma_n \sigma_n^{-2}$. This implies that the branched cover of L_{stab} is given by the union of u_n^{ot} and $\Omega(L)$. Untwisting the braid $\sigma_1 \cdots \sigma_{n-1} \sigma_n$ as explained in Remark 3.3 shows that u_n^{ot} and $\Omega(L)$ do not link.

To demonstrate that the contact manifold represented by u_n^{ot} is an overtwisted 3-sphere, we first use Kirby calculus to see that the underlying smooth manifold is S^3. Using formula (2.3), we compute $d_3 = -\frac{1}{2} + p - 1$. (We have $c_1(s_J) = 0$, $\text{sign}(X) = 0$, $\chi(X) = 1 + 2(p-1)$.) Since we know that ξ_{std} is the unique tight contact structure on S^3, and $d_3(\xi_{std}) = -\frac{1}{2}$, it follows that the contact structure given by the diagram u_n^{ot} is overtwisted. The branched cover of L_{stab} is then the connected sum of this overtwisted sphere and the branched cover of L. ∎

Corollary 3.7.   Suppose that $L_n = \sigma_1 \cdots \sigma_{n-1} b \in B_n$ is an n-braid, and $L_{n+1} = \sigma_1 \cdots \sigma_{n-1} b \in B_{n+1}$ is an $(n + 1)$-braid obtained from L_n by an addition of a trivial $(n + 1)$th strand. Then the branched cover of L_{n+1} can be described by the surgery diagram, which is the union of $\Omega(L)$ and u_n^-, where u_n^- *does not* link $\Omega(L)$. □
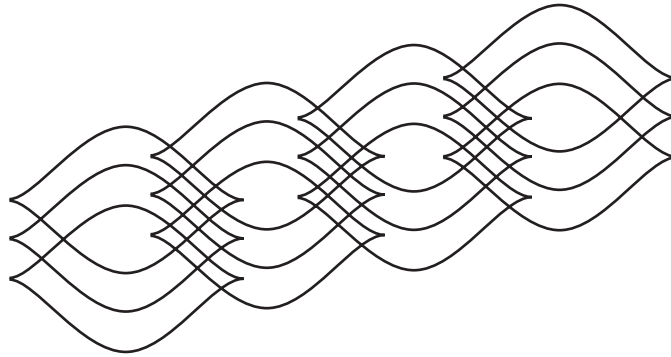
Fig. 15.   A Legendrian surgery diagram for $\Sigma_5(\sigma_1^4)$.

Proof.   This follows from the identity $L_{n+1} = \sigma_1 \cdots \sigma_{n-1}\sigma_n\sigma_n^{-1}b$. The word b does not contain $\sigma_n^{\pm n}$, so untwisting the unknot $\sigma_1 \cdots \sigma_{n-1}\sigma_n$ as in Remark 3.3, we get unlinked diagrams $\Omega(L)$ and u_n^-.

   We also observe that on the level of contact manifolds, we are taking a connected sum with $\#_p S^1 \times S^2$, where the latter is equipped with its unique Stein fillable contact structure. ∎

   It is now easy to obtain surgery diagrams of all p-fold branched covers of 2-braids.

Example 3.8.   A surgery diagram for the 5-fold cover of the transverse braid $(\sigma_1)^4$ is shown in Figure 15. □

Remark 3.9.   Even though every closed n-braid is isotopic to a braid containing a string $\sigma_1\sigma_2\sigma_3 \ldots \sigma_{n-1}$, we may want to start with an open book corresponding to another version of transverse unknot, say $\sigma_2\sigma_1\sigma_3 \cdots \sigma_{n-1}$ This will be useful in Section 5.2. To obtain this other open book, we consider Figure 10 and change the curves $\alpha_1^2, \alpha_2^2, \ldots, \alpha_{p-1}^2$, so that they now go through the top three rows of the grid-like page. This is shown on Figure 16; the other curves $\alpha_1^k, \alpha_2^k, \ldots, \alpha_{p-1}^k$ for $k \neq 2$ remain the same. For open books, this change corresponds to plumbing positive Hopf bands together in a slightly different way to form the same page. Analyzing the push-offs of the curves α_j^k as in Proposition 3.2, we see that the monodromy of the open book can now be expressed as $(D_1^{n-1} \circ \cdots \circ D_{p-1}^{n-1}) \circ \cdots \circ (D_1^1 \circ \cdots \circ D_{p-1}^1) \circ (D_1^2 \circ \cdots \circ D_{p-1}^2)$, which by Lemma 3.1 corresponds to the braid $\sigma_2\sigma_1\sigma_3 \cdots \sigma_{n-1}$ as required.
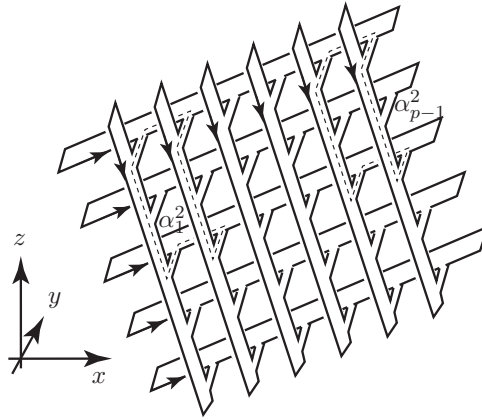
Fig. 16.   A different choice of curves α_j^2 produces an open book whose monodromy is $\sigma_2\sigma_1\sigma_3\cdots\sigma_{n-1}$.

Another case worth mentioning is the initial unknot given by the braid $\sigma_n\cdots\sigma_2\sigma_1$. In this case, we have the same open book as for the unknot $\sigma_1\sigma_2\cdots\sigma_n$, with the role of the curve α_j^k played by α_j^{n-k}.

In principle, it is not necessary to single out the braid word that gives the unknot: we can as well start from the trivial braid and obtain $(\Sigma_p(L),\xi_p(L))$ as a result of surgery on $\#_{p-1}S^1\times S^2$. However, the presence of 1-handles seems to complicate matters. $\square$

4   Properties of Branched Covers

In this section, we prove Theorems 1.3 and 1.4. The proofs are very similar to those of [22, Sections 4 and 5].

4.1   Quasipositive braids and stabilizations

Recall [23] that a braid is called *quasipositive* if its braid word is a product of conjugates of the standard generators.

Proof of Theorem 1.3.   If L is quasipositive, we can resolve a few positive crossings to convert the braid representing L into a braid equivalent to a trivial one (of the same braid index). The p-fold cover branched over the trivial braid is a connected sum of several copies of $(S^1\times S^2,\xi_0)$, which is Stein fillable (ξ_0 here stands for the unique Stein fillable contact structure on $S^1\times S^2$). Putting the positive crossings back in, by Lemma 3.1 we see that the monodromy of the open book for $(\Sigma_p(L),\xi_p(L))$ is given by a composition of

positive Dehn twists. It follows that the contact manifold is Stein fillable. The second part of the theorem follows from Corollary 3.6. ∎

4.2 Homotopy invariants

Proof of Theorem 1.4.   The fact that $c_1(\mathfrak{s}_L) = 0$ follows immediately: $\mathfrak{s}_\xi$ is the restriction to Y of the Spin^c structure $\mathfrak{s}_J$ described in Section 2.3; $c_1(\mathfrak{s}_J)$ evaluates as 0 on each homology generator corresponding to either a (-1) or a $(+1)$ surgery, because all surgeries are performed on standard Legendrian unknots with rotation number 0.

For the second part of the theorem, suppose that two closed braids L and L' are isotopic as smooth knots, and that $sl(L) = sl(L')$. By the Markov theorem for smooth knots [3], L' can be obtained from L by a sequence of braid isotopies and (positive and negative) braid stabilizations and destabilizations. Braid isotopies and positive stabilizations preserve both sl and the d_3 invariant, since they do not change the transverse link type. Each negative stabilization (resp. destabilization) decreases (resp. increases) the self-linking number by 2 and the d_3 invariant by $p - 1$, since, as we saw in Corollary 3.6, transverse stabilization gives the connected sum with the overtwisted sphere in Figure 14. But if $sl(L) = sl(L')$, every negative stabilization must be compensated by a negative destabilization. It follows that $d_3(\xi_L) = d_3(\xi_{L'})$. ∎

Corollary 4.1.   Fix $p \geq 2$. Let T be a transverse link smoothly isotopic to the (m, n) torus link, $m, n > 0$. The branched cover $\Sigma_p(T)$ is then the Brieskorn sphere $\Sigma(m, n, p)$. If $sl(T) = sl_{max} = mn - m - n$, then $\xi_p(T)$ is Stein fillable. Otherwise $\xi_p(T)$ is overtwisted. For different values of $sl(T)$, these overtwisted structures have different d_3 invariants. □

Proof.   By classification of transverse torus links [8], when T has the maximal self-linking number, it has a positive braid representation. Otherwise, T is transversally isotopic to the maximal one after transversely destabilized $(sl_{max} - sl(T))$ times. ∎

4.3 Overtwisted branched covers

We generalize the second part of Theorem 1.3, and show that the branched covers are overtwisted for a large family of transverse links.
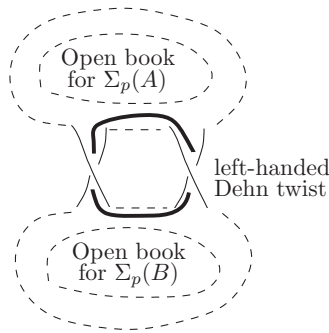
Fig. 17.   The plumbed sum of the open books $(\tilde{D}, \tilde{\phi}_A)$, $(\tilde{D}, \tilde{\phi}_B)$, and a left-handed Hopf band.

Proposition 4.2.   Suppose that the transverse link L is represented by a closed braid such that its braid word ϕ_L contains a factor of σ_i^{-1} but no σ_i's for some $i > 0$. (This means that all the crossings in the braid diagram on the level between the ith and the $(i + 1)$th strands are negative.) Then the branched p-fold cover $\Sigma_p(L)$ is overtwisted for any $p \geq 2$.   □

Proof.   We will use the right-veering monodromy criterion for tightness [17]. We refer the reader to [17] for precise definitions; informally, the right-veering property of the open book (S, ϕ) means that every properly embedded arc on S maps "to the right" of itself under ϕ. A basic example of right-veering monodromy is given by a right-handed Dehn twist, Figure 1. Theorem 1.1 of [17] says that a contact structure is tight if and only if every compatible open book has right-veering monodromy. We will use the "only if" part: if the monodromy of an open-book decomposition is not right-veering, then the contact structure is overtwisted. (This is in fact the criterion for overtwistedness that was first given in [15] in terms of "sobering arcs".) We also recall that the composition of two right-veering monodromies is right-veering.

To prove the proposition, observe that the negative crossings between ith and $(i + 1)$th strands yield the open book $(\tilde{D}, \tilde{\phi}_L)$ such that the Dehn twists about curves α_j^i are all left-handed for $j = 1, \ldots, p - 1$.

Remove from L all the (negative) crossings between the ith and the $(i + 1)$th strands (in other words, remove all the negative factors of σ_i from the braid word ϕ_L). The link L then splits into two links A and B such that the corresponding braid words ϕ_A and ϕ_B contain only generators σ_j with $j < i$ resp. $j > i$. Let $(\tilde{D}, \tilde{\phi}_A)$, $(\tilde{D}, \tilde{\phi}_B)$ be the open books for the branched covers $\Sigma_p(A)$ and $\Sigma_p(B)$, and consider the plumbed sum of $(\tilde{D}, \tilde{\phi}_A)$, $(\tilde{D}, \tilde{\phi}_B)$, and a left-handed Hopf band. The resulting open book is shown on Figure 17, and is in fact a negative stabilization of the connected sum $(\tilde{D}, \tilde{\phi}_A)\#(\tilde{D}, \tilde{\phi}_B)$. It is easy to

see that the monodromy of a negatively stabilized open book is not right-veering. The open book $(\tilde{D}, \tilde{\phi}_L)$ is obtained from this non-right-veering open book by a sequence of negative stabilizations and additional left-handed Dehn-twists, and so cannot be right-veering either (because a composition of a right-handed Dehn twist and a right-veering monodromy is right-veering). ∎

Remark 4.3. If the branched p-fold covers of two transverse links L_1 and L_2 of the same topological type are both overtwisted and $sl(L_1) = sl(L_2)$, then Theorem 1.4 together with Eliashberg's classification of overtwisted contact structures implies that $\Sigma_p(L_1)$ is contactomorphic to $\Sigma_p(L_2)$. □

We therefore have

Corollary 4.4. In Table 1 in [5] of transverse knots, all pairs (except perhaps for the representatives of the knot 11_{a240}) give rise to contactomorphic p-fold branched covers for all $p \geq 2$. □

In view of the previous remark, showing that certain branched covers are overtwisted can be useful. We thus illustrate two other ways to establish overtwistedness (our examples below are all included in Proposition 4.2, but the methods can be used for other links as well).

The first method applies in the rare cases where the classification of tight contact structures is known for the smooth manifold $\Sigma_p(L)$. For example, this is the case for double covers of 2-bridge links: it is well known that these are lens spaces, and the tight contact structures on lens spaces were classified in [16].

Consider the transverse 2-braid $L = \sigma_1^{-k}$ where $k \geq 1$; its branched double cover is the lens space $-L(k, 1) = L(k, k-1)$, with the contact structure ξ given by the surgery diagram on Figure 18 (where $(+1)$ contact surgery is performed on each of $k + 1$ successive push-offs of the Legendrian unknot of $tb = -1$). We compute the d_3 invariant of this contact structure. If X is the 4-manifold corresponding to the surgery, then $\mathrm{sign}(X) = k - 1$ (indeed, the intersection form for X has zeroes on the diagonal and -1's for all other entries; it is easy to see that the matrix has an eigenvalue 1 of order k and an eigenvalue $-k$ of order 1). We also have $c_1(X) = 0$, and $\chi(X) = k + 2$. Therefore, from (2.3) we obtain $d_3(\xi) = \frac{-5k-1}{4}$. On the other hand, by [16], the lens space $L(k, k-1)$ carries a unique tight contact structure ξ_0; this contact structure is the boundary of a linear plumbing (also shown on Figure 18). The corresponding Stein 4-manifold X_0 has $c_1(X_0) = 0$, $\mathrm{sign}(X_0) = 0$
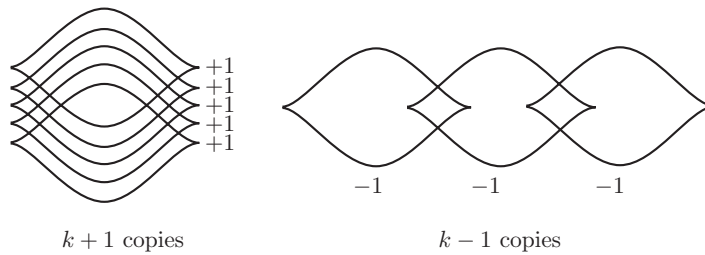
Fig. 18. The branched double cover of σ_j^{-k} (left) and the unique tight contact structure on $L(k, k - 1)$ (right).
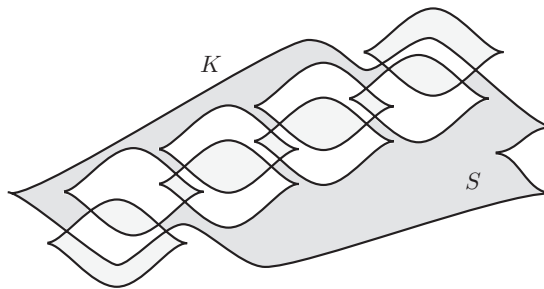

Fig. 19. The Legendrian unknot knot K bounds an overtwisted disk in the surgered manifold.

and $\chi(X_0) = k$, so $d_3(\xi_0) = -\frac{k}{2}$. It follows that the contact structure ξ is not isotopic to ξ_0, and therefore must be overtwisted.

Another way to prove overtwistedness is simply to find an overtwisted disk in the surgery diagram. Admitting that these pictures get unwieldy even for simple links, we exhibit such a disk for the overtwisted sphere u^{ot} described in Figure 14 (i.e. the branched p-fold cover of σ^{-1}). Indeed, the surface S shown on Figure 19 induces the 0-framing on each component of the surgery link u^{ot}, and the (-2)-framing on the Legendrian knot K. (We assume that all Legendrian knots are oriented as the boundary of S). Since $(+1)$-contact surgery is 0-framed Dehn surgery, S becomes a disk bounded by K in the surgered manifold. Then the equality $tb(K) = $ the surface framing of $K = -2$ implies that this disk is overtwisted.

5   Can We Distinguish Transverse Knots?

We can now use the constructions from previous sections to examine the branched covers of certain transverse knots and prove Theorem 1.2 (see Corollaries 5.5, 5.6, and Theorem 5.9). We already saw that for most examples of [5], the branched covers do not

detect the difference between transverse knots. We now consider the remaining pairs of nonisotopic transverse knots with the same classical invariants from [4, 5, 19], and try to distinguish them via the corresponding contact structures.

5.1   Birman–Menasco examples

The methods of Birman and Menasco [4, 5] produce examples that are pairs of 3-braids L_1, L_2 related by a *negative flype*. This means that $L_1 = \sigma_1^u \sigma_2^v \sigma_1^w \sigma_2^{-1}$ and $L_2 = \sigma_1^u \sigma_2^{-1} \sigma_1^w \sigma_2^v$.

Recall that the contact structure $\bar{\xi}$ *conjugate to* ξ is obtained from ξ by reversing the orientation of contact planes.

Proposition 5.1.   Transverse 3-braids L_1 and L_2 related by a negative flype give rise to conjugate contact structures on the branched covers: $\xi_p(L_1)$ is contactomorphic to $\bar{\xi}_p(L_2)$.

$\square$

Proof.   We write the closed braids as

$$L_1 = (\sigma_1 \sigma_2) \sigma_2^{v-1} \sigma_1^w \sigma_2^{-1} \sigma_1^{u-1}, \qquad L_2 = (\sigma_2 \sigma_1) \sigma_1^{u-1} \sigma_2^{-1} \sigma_1^w \sigma_2^{v-1}.$$

Observe that L_2 can be taken to $(\sigma_1 \sigma_2) \sigma_2^{u-1} \sigma_1^{-1} \sigma_2^w \sigma_1^{v-1}$ by a transverse isotopy. Using the method in Theorem 3.4 and the following corollaries, we can draw surgery diagrams for the branched covers of L_1 and L_2. For example, double covers for the case where $u-1, v-1, w \geq 1$ are shown on Figure 20 (top); we see that they are obtained by contact surgeries on two links that are Legendrian mirrors of one another. Similarly, p-fold branched covers for L_1 and L_2 are also obtained by surgery on Legendrian mirrors, since the corresponding diagrams are obtained by taking $(p-1)$ copies of the surgery link for the double cover linked as dictated by Figure 11. For example, the triple cover for L_2 is shown on Figure 20 (bottom). For negative u, v, or w the pictures are similar; besides, the case $v \leq 0$ is already covered by Corollary 4.4.

We have shown that the surgery link diagram $\Omega_p(L_2)$ for L_2 is the Legendrian mirror of the link $\Omega_p(L_1)$ for L_1. Now, observe that one link is taken to the other by the map $(x, y, z) \mapsto (-x, y, -z)$. This map reverses the sign of the standard contact form $dz - y\,dx$ (i.e. the orientation of contact planes on S^3) and extends to the map of branched covers that takes $\xi_p(L_1)$ to $\bar{\xi}_p(L_2)$.   ∎
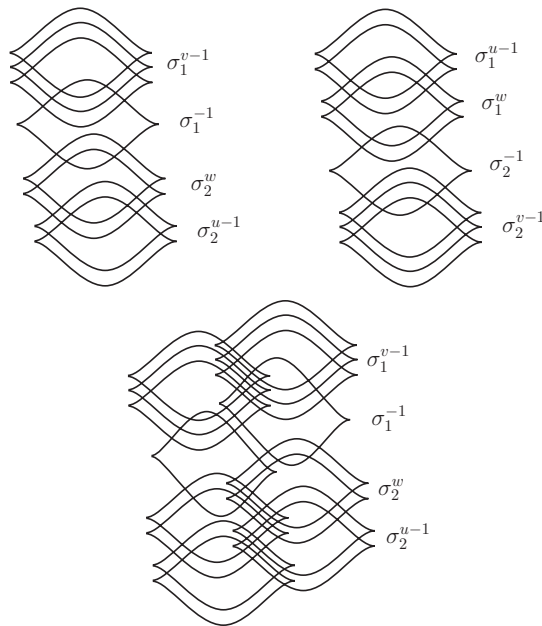
Fig. 20.   Branched double covers of L_1 (top left) and L_2 (top right). We assume that $u - 1, v - 1, w \geq 1$; a $(+1)$ contact surgery is performed on unknots marked with $+1$, Legendrian surgery on all other components. A diagram for the branched 3-fold cover of L_2 is shown at the bottom; branched 3-fold cover of L_1 is obtained by contact surgery on a mirror of this link. To obtain p-fold covers, take $(p - 1)$ copies of the surgery link for the double cover linked in a way similar to the 3-fold cover case.

Remark 5.2.   Alternatively, the previous proposition can be proved by using open books. A careful examination of the monodromy shows that the open book for $(\Sigma_p(L_2), \xi_p(L_2))$ can be obtained from the open book for $(\Sigma_p(L_1), \xi_p(L_1))$ by reversing the orientation of the pages as well as the orientation of the S^1 direction in the mapping torus. This operation preserves the orientation of the 3-manifold but reverses the orientation of contact planes.

$\square$

We can generalize Proposition 5.1 as follows.

Proposition 5.3.   Let the braid L_2 be obtained by reading the braid word L_1 backward, i.e. if $L_1 = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{k-1}} \sigma_{i_k}$ then $L_2 = \sigma_{i_k} \sigma_{i_{k-1}} \cdots \sigma_{i_2} \sigma_{i_1}$. Then the contact structures $\xi_p(L_1)$ and $\xi_p(L_2)$ are conjugate to one another for any $p \geq 2$. $\square$

Proof.   Write

$$L_1 = (\sigma_1 \sigma_2 \cdots \sigma_{n-1}) \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_{l-1}} \sigma_{j_l}.$$

Then the braid word for L_2 is conjugate to

$$L_2 = (\sigma_{n-1} \cdots \sigma_2 \sigma_1)\sigma_{j_l}\sigma_{j_{l-1}} \cdots \sigma_{j_2}\sigma_{j_1}.$$

In the surgery diagram for cover of L_1, the part of the surgery link corresponding to σ_{j_r} will be above that for σ_{j_s} when $r < s$; for cover of L_2, it will be below. In both cases, the surgery unknots corresponding to σ_{j_r} and σ_{j_s} with $r < s$ will be linked (in exactly the same way) iff $j_r \leq j_s$; using the braids-to-surgeries description from Section 3.2, cf. Figure 11, we see that in fact the surgery links for the two branched covers are Legendrian mirrors of one another. It follows that the resulting contact structures $\xi_p(L_1)$ and $\xi_p(L_2)$ are conjugate to one another.

(Alternatively, we could rotate L_2 to get

$$L_2 = (\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1})\sigma_{n-j_l}\sigma_{n-j_{l-1}} \cdots \sigma_{n-j_2}\sigma_{n-j_1},$$

and draw the surgery diagrams similar to the Birman–Menasco braids in Proposition 5.1.) ∎

Proposition 5.4.   For any transverse link L, $p \geq 2$, the contact structure $\xi_p(L)$ is isomorphic to its conjugate $\bar{\xi}_p(L)$.   □

Proof.   We need to find an involution of the smooth manifold $\Sigma_p(L)$ that induces the orientation reversal on contact planes. For a page P of the open book described in Lemma 3.1, there is an orientation-reversing map $I : P \to P$ that maps kth sheet to the $(p + 1 - k)$th sheet, acting as a reflection, and takes the curve α_k^j to the curve α_{p-k}^{n-j} (see Figure 21). If p is odd, the $(p+1)/2$-th sheet is mapped to itself, and if n is even, the curve $\alpha_{(p+1)/2}^{n/2}$ is mapped to itself. Moreover, $(D_k^j)^{-1}I = ID_{p-k}^{n-j}$, i.e. the involution I takes right-handed Dehn twists to the left-handed ones. If $\tilde{\sigma}_j$ is the lift of the half-twist σ_j as in Figure 2, we have

$$(\tilde{\sigma}_j)^{-1}I = \left(D_{p-1}^j\right)^{-1}\cdots\left(D_2^j\right)^{-1}\left(D_1^j\right)^{-1}I = ID_1^{n-j}D_2^{n-j}\cdots D_{p-1}^{n-j} = I\tilde{\sigma}_{n-j}.$$

Write ϕ_L for the braid word for L, and let $\phi_{L'}$ be the braid word obtained by changing every half-twist generator σ_j to σ_{n-j}. The braids ϕ_L and $\phi_{L'}$ are related by a braid isotopy (rotating the braid), so if L' is the transverse link corresponding to the braid $\phi_{L'}$, then L
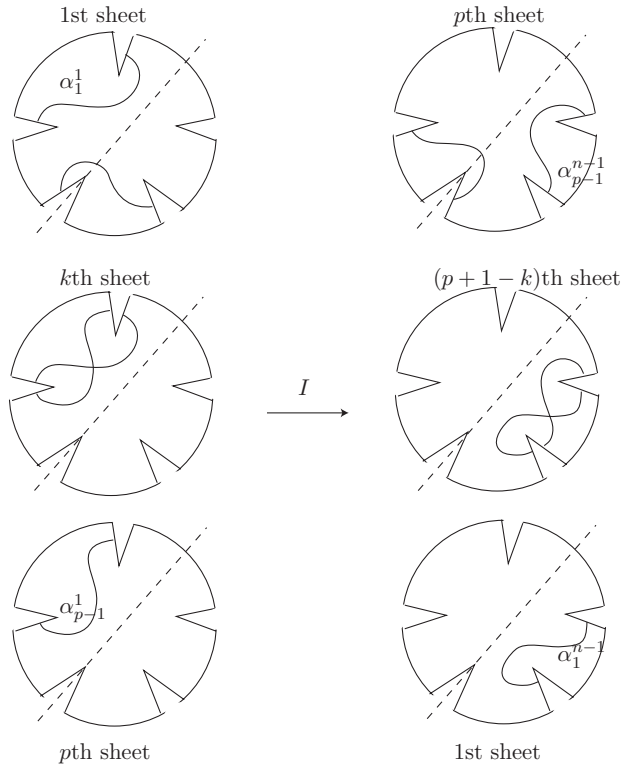
Fig. 21. The involution I on a page of the open book.

and L' are transversely isotopic. However, we have

$$(\tilde{\phi}_L)^{-1}I = I\tilde{\phi}_{L'}.$$

If we extend the map I to an orientation-preserving map $R : P \times [0, 1] \to P \times [0, 1]$, defined by $R(x, t) = (I(x), 1 - t)$, R descends to open books, taking the open book (P, ϕ_L) to the open book $(-P, (\phi_{L'})^{-1})$. The latter open book is compatible with the contact structure $\bar{\xi}_{L'}$, which is isotopic to $\bar{\xi}_L$. It follows that ξ_L and $\bar{\xi}_L$ are isomorphic. $\blacksquare$

The last two propositions apply in the following special cases, proving Theorem 1.2.

Corollary 5.5. Let L be a Legendrian link, $\bar{L}$ its Legendrian mirror, and consider the transverse push-offs L^+ and $\bar{L}^-$. Then the corresponding p-fold branched covers are contactomorphic for all p. $\square$
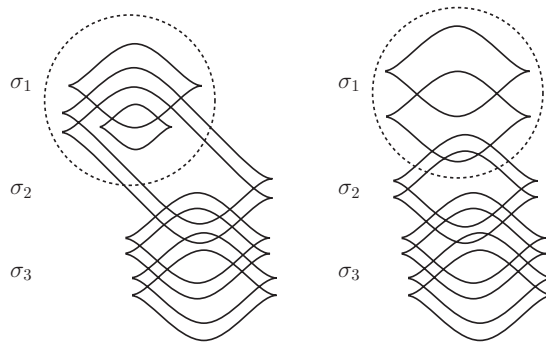
Fig. 22.   Branched double covers of L_1^+ (left) and L_2^+ (right).

Corollary 5.6.   If 3-braids L_1 and L_2 are related by a negative flype, then $\xi_p(L_1)$ and $\xi_p(L_2)$ are isomorphic. $\square$

Remark 5.7.   Double branched covers of the Birman–Menasco 3-braids were studied in [22]. $\square$

5.2   Ng–Ozsváth–Thurston examples

In [19], transverse knots are given as push-offs of Legendrian knots, and the latter are represented by grid diagrams of their (smooth) mirrors. We recall how to obtain a positive transverse push-off of a Legendrian knot given by such a grid diagram (cf. [19]). First, let the horizontal segments in the diagram go over the vertical segments (this is opposite to the convention for grid diagrams and produces a front projection for the Legendrian knot). Then keep every vertical segment oriented upward (i.e. has O above X), and replace every vertical segment oriented downward by the complementary vertical segment. The result is a braid that goes from the bottom to the top of the diagram and represents the positive push-off of the given Legendrian knot. To obtain the braid for the negative transverse push-off, reverse the orientation of the Legendrian knot (by replacing O's by X's and vice versa in the grid diagram), and repeat the above procedure.

We consider transverse push-offs L_1^+ and L_2^+ of the Legendrian representatives of the pretzel knot $P(-4, -3, 3)$ [19, Figure 4]. These are given by transverse closed braids

$$L_1^+ = \sigma_3^{-1}\sigma_2\sigma_3\sigma_1\sigma_1\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-2} \quad \text{and} \quad L_2^+ = \sigma_3\sigma_2\sigma_1\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-2}\sigma_3.$$
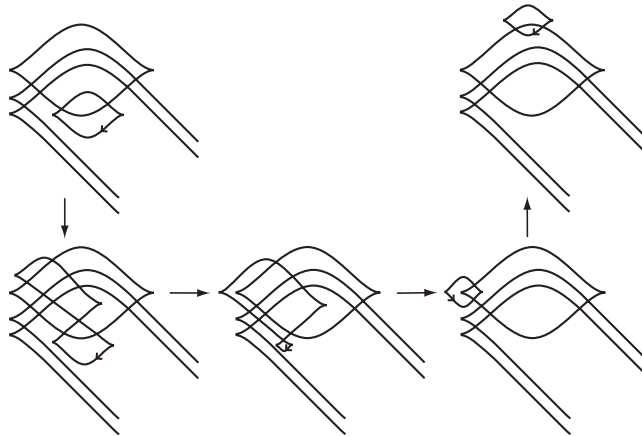
Fig. 23.   The Legendrian links in Figure 22 can be related by Legendrian Reidemeister moves.

A braid isotopy takes these braids to

$$L_1^+ = (\sigma_2\sigma_1\sigma_3)\sigma_3\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-2}\sigma_1^{-1} \qquad \text{and} \qquad L_2^+ = (\sigma_1\sigma_2\sigma_3)\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-2}\sigma_1.$$

We can now draw surgery diagrams for the double branched covers of L_1^+ and L_2^+; they are shown on Figure 22. Recall Remark 3.9 and Figure 16. Note that the two surgery links differ only in the circled region; this corresponds to the fact that the braids for L_1^+ and L_2^+ differ only by exchanging two generators σ_1^{-1} and σ_1 (together with a different choice of the open book). We observe that the surgery links are in fact Legendrian isotopic. The isotopy can be performed via a sequence of Legendrian Reidemeister moves indicated on Figure 23.

The transverse push-offs $(L_1')^+$ and $(L_2')^+$ of the Legendrian representatives of the pretzel knot $P(-6, -3, 3)$, [19, Figure 5] can be treated in the same way. Indeed, these are given by braids

$$(L_1')^+ = \sigma_4^{-1}\sigma_3\sigma_4\sigma_2\sigma_1\sigma_4\sigma_2\sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1},$$
$$(L_2')^+ = \sigma_4\sigma_3\sigma_2\sigma_1\sigma_4^{-1}\sigma_2\sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_4$$

braid isotopic to

$$(L_1')^+ = (\sigma_2\sigma_1\sigma_3\sigma_4)\sigma_1\sigma_3\sigma_4\sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_1^{-1},$$
$$(L_2')^+ = (\sigma_1\sigma_2\sigma_3\sigma_4)\sigma_1^{-1}\sigma_3\sigma_4\sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_1.$$

As in the previous example, we make a different choice of the initial unknot, and then switch the two factors of σ_1 and σ_1^{-1} to relate the braids. The surgery diagrams are very similar to the previous case; the surgery links have more surgery components, but differ only in the circled region exactly as above, and can be related by a sequence of Reidemeister moves.

It is conjectured in [19] that all pretzel knots $P(-2n, -3, 3)$ are not transversely simple, and if L_1^n, L_2^n are the Legendrian representatives of $P(-2n, -3, 3)$ similar to those considered above, then $(L_1^n)^+$ and $(L_2^n)^-$ are not transversely isotopic. Our argument, however, clearly generalizes to show that the corresponding branched double covers are contactomorphic.

Moreover, our argument for the knots L_1^+ and L_2^+ will work for any two braids of the form

$$K_1 = \sigma_1^m\sigma_2\sigma_1^{-1}w \qquad \text{and} \qquad K_2 = \sigma_1^{-1}\sigma_2\sigma_1^m w,$$

where w is any braid word on generators $\sigma_2, \ldots, \sigma_{n-1}$, and $m > 0$. Indeed, such two closed braids are isotopic to

$$K_1 = (\sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1})\sigma_1^{-1}w'\sigma_1^{m-1} \qquad \text{and} \qquad K_2 = (\sigma_2\sigma_1\sigma_3\cdots\sigma_{n-1})\sigma_1^{m-1}w'\sigma_1^{-1}$$

and the corresponding surgery diagrams differ by the same local change as above, except that instead of the single Legendrian unknot to be moved we have $(m - 1)$ copies of its Legendrian push-offs. A similar sequence of Reidemeister moves can be used to perform this local change. We observe that two such braids are in fact related by a negative flype. We thus have

Proposition 5.8. Let K_1, K_2 be two transverse braids related by the special kind of negative flype satisfying

$$K_1 = \sigma_1^m\sigma_2\sigma_1^{-1}w, \qquad K_2 = \sigma_1^{-1}\sigma_2\sigma_1^m w,$$

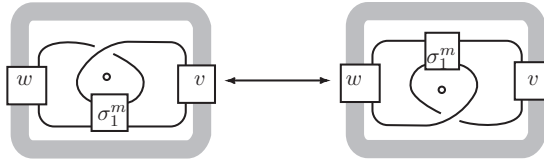**Fig. 24.** A negative flype move. The gray band means nonbraided $(n-2)$ strands.

where w is a word in $\sigma_2, \ldots, \sigma_{n-1}$, and m is an integer. Then the branched double covers of K_1 and K_2 are contactomorphic. $\qquad\square$

Proof. The case $m > 0$ is considered above. When $m \leq 0$, the p-fold cyclic branched covers for K_1 and K_2 of any p are overtwisted by Proposition 4.2. Since they share the homotopy invariants, thus they are contactomorphic. $\qquad\blacksquare$

More generally we have the following.

Theorem 5.9. Suppose K_1 and K_2 are related each other by a negative flype move sketched in Figure 24, i.e.

$$K_1 = \sigma_1^m v \sigma_1^{-1} w, \qquad K_2 = \sigma_1^{-1} v \sigma_1^m w,$$

where v and w are any braid words in generators $\sigma_2, \ldots, \sigma_{n-1}$ and $m \in \mathbb{Z}$. Then the branched double covers $(\Sigma_2(K_1), \xi_2(K_1))$ and $(\Sigma_2(K_2), \xi_2(K_2))$ are contactomorphic. $\qquad\square$

Proof. Consider positive stabilizations of K_1 and K_2. Since a positive stabilization preserves transverse knot type, we use the same notations K_1, K_2. Let v' (resp. w') be the braid words in $\sigma_3, \ldots, \sigma_n$ obtained from v (resp. w) by translation $\sigma_k \mapsto \sigma_{k+1}$. Then we have

$$
\begin{aligned}
K_1 &= \sigma_1^m \sigma_1^{-1} \sigma_1 v \sigma_1^{-1} w && \text{isotopy} \\
&= \sigma_2^m \sigma_2^{-1} \underline{\sigma_1} \sigma_2 v' \sigma_2^{-1} w' && \text{(+)stabilization} \\
&= \sigma_2 \sigma_1^m v' \sigma_2 \sigma_1^{-1} \sigma_2^{-1} w' && \text{isotopy} \\
&= \sigma_2 v' \underline{\sigma_1^m \sigma_2 \sigma_1^{-1}} \sigma_2^{-1} w' && \text{isotopy.}
\end{aligned}
$$

Similarly, we have

$$K_2 = \sigma_2 \sigma_1^{-1} v' \sigma_2 \sigma_1^m \sigma_2^{-1} w' = \sigma_2 v' \underline{\sigma_1^{-1} \sigma_2 \sigma_1^m} \sigma_2^{-1} w'.$$
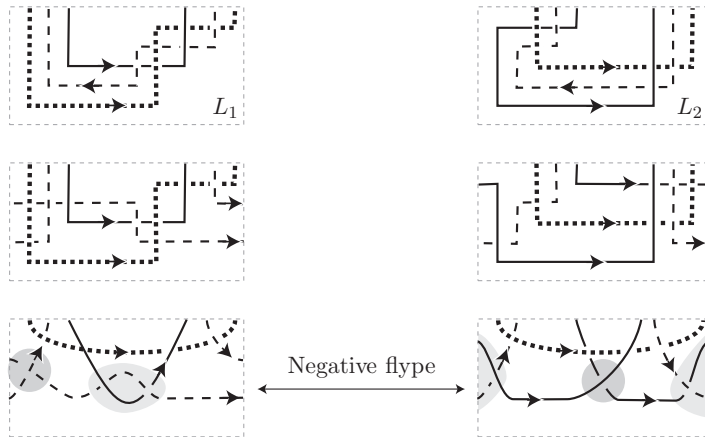
Fig. 25.   Right and left sides are identified on each picture.
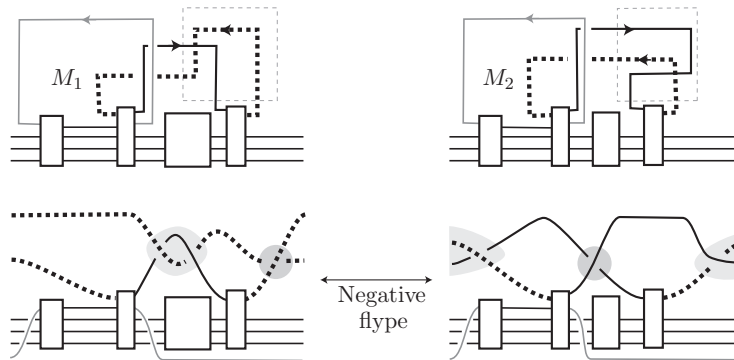

Fig. 26.   Right and left sides are identified on each picture.

Thus they satisfy the condition of Proposition 5.8.   ■

Example 5.10.   Let L_1, L_2 (resp. L_1', L_2') be the Legendrian $m(10_{132})$ (resp. $m(12n_{200})$) knots studied in [19]. Let M_1, M_2 be the Legendrian $(2,3)$-cables of the $(2,3)$-torus knot found in [11, 18]. The positive push-offs of every pair satisfy the condition of Theorem 5.9. Therefore, double branched covers for each pair are contactomorphic.   □

Proof.   The front projections of L_1, L_2 (resp. L_1', L_2') given in [19, Figures 2, 3] only differ in the dashed boxes shown in Figure 25; we then see that the corresponding closed braids representing L_1^+, L_2^+ (resp. $(L_1')^+, (L_2')^+$) are related to each other by a negative flype.

Similarly, the closed braids $(M_1)^+, (M_2)^+$ only differ in the dashed boxes sketched in Figure 26 and are also related to each other by a negative flype. Here we use the Legendrian fronts for M_1, M_2 given in [19].   ■

Acknowledgments

References

[1]   Akbulut, S., and B. Ozbagci. "Lefschetz fibrations on compact Stein surfaces." *Geometry & Topology* 5 (2001): 319–34.

[2]   Bennequin, D. "Entrelacements et équations de Pfaff." *Astérisque* 107–108 (1983): 87–161.

[3]   Birman, J. *Braids, Links and Mapping Class Groups*. Annals of Mathematics Studies 82. Princeton: Princeton University Press, 1974.

[4]   Birman, J., and B. Menasco. "Stabilization in the braid groups 2: Transversal simplicity of knots." *Geometry & Topology* 10 (2006): 1425–52.

[5]   Birman, J., and B. Menasco. "A note on transversal knots that are closed 3-braids." (2007): preprint math.GT/0703669.

[6]   Ding, F., and H. Geiges. "A Legendrian surgery presentation of contact 3-manifolds." *Mathematical Proceedings of the Cambridge Philosophical Society* 136, no. 3 (2004): 583–98.

[7]   Ding, F., H. Geiges, and A. Stipsicz. "Surgery diagrams for contact 3-manifolds." *Turkish Journal of Mathematics* 28, no. 1 (2004): 41–74.

[8]   Etnyre, J. "Transversal torus knots." *Geometry & Topology* 3 (1999): 253–68.

[9]   Etnyre, J. *Legendrian and Transversal Knots*, 105–85. Handbook of Knot Theory. Amsterdam, the Netherlands: Elsevier, 2005.

[10]   Etnyre, J. "Lectures on Open Book Decompositions and Contact Structures." In *Floer Homology, Gauge Theory, and Low-Dimensional Topology*, 103–41. Clay Mathematics Proceedings 5. Providence, RI: American Mathematical Society, 2006.

[11]   Etnyre, J., and K. Honda. "Cabling and transverse simplicity." *Annals of Mathematics* 162, no. 3 (2005): 1305–33.

[12]   Farb, B., and D. Margalit. "A Primer on Mapping Class Groups." Version 2.95. http://www.math.utah.edu/~margalit/primer.

[13]   Giroux, E. "Géométrie de contact: de la dimension trois vers les dimensions supérieures." In *Proceedings of the International Congress of Mathematicians*, 405–14. Vol. 2. Beijing: Higher Education Press, 2002.

[14]   Gompf, R. "Handlebody construction of Stein surfaces." *Annals of Mathematics* 148, no. 2 (1998): 619–93.

[15]   Goodman, N. "Overtwisted open books from sobering arcs." *Algebraic & Geometric Topology* 5 (2005): 1173–95.

[16]   Honda, K. "On the classification of tight contact structures 1." *Geometry & Topology* 4 (2000): 309–68.

[17]   Honda, K., W. Kazez, and G. Matić. "Right-veering diffeomorphisms of compact surfaces with boundary 1." *Inventiones Mathematicae* 169, no. 2 (2007): 427–49.

[18]   Menasco, W., and H. Matsuda. "An addendum on iterated torus knots." (2006): preprint math.GT/0610566.

[19]   Ng, L., P. Ozsváth, and D. Thurston. "Transverse knots distinguished by knot Floer homology." *Journal of Symplectic Geometry* (forthcoming)

[20]   Ozsváth, P., Szabó, Z., and D. Thurston. "Legendrian knots, transverse knots and combinatorial Floer homology." *Geometry & Topology* 12, no. 2 (2008): 941–80.

[21]   Plamenevskaya, O. "Contact structures with distinct Heegaard Floer invariants." *Mathematical Research Letters* 11, no. 4 (2004): 547–61.

[22]   Plamenevskaya, O. "Transverse knots, branched double covers and Heegaard Floer contact invariants." *Journal of Symplectic Geometry* 4, no. 2 (2006): 149–70.

[23]   Rudolph, L. "Quasipositivity as an obstruction to sliceness." *Bulletin of the American Mathematical Society* 29, no. 1 (1993): 51–9.