Fractal nature of the space of knotted curves

Monash University Colloquium

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Recall, the 3-dimensional sphere is

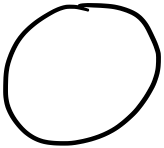
$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$$
$$= \mathbb{R}^3 \cup \{\infty\}$$

<u>Def</u>: A <u>Knot</u> is a smooth embedding

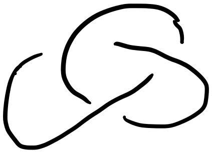
$$f : S^1 \hookrightarrow S^3$$

where $S^1 = \{z \mid |z|^2 = 1\} \subseteq \mathbb{C}$ is the unit circle.

Exs:



unknot



trefoil

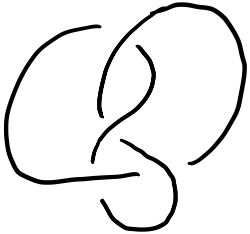
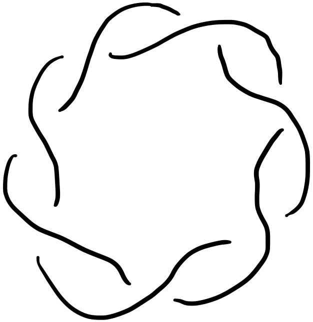
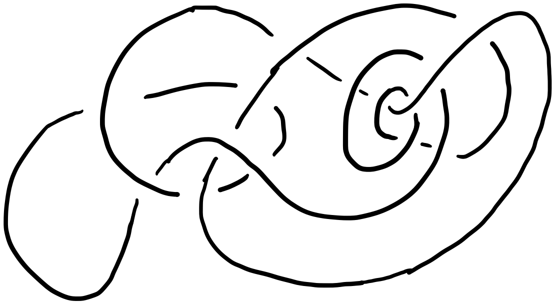


figure-eight





Knots can arise from singularities.

Ex: Let C be the complex curve defined by

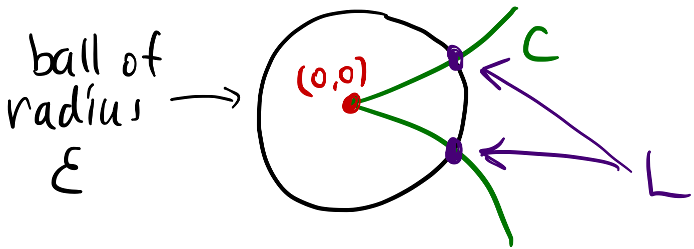
$$z^2 - w^3 = 0$$

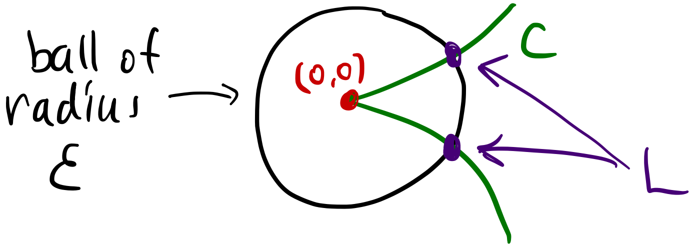
It has a singularity at $(z,w) = (0,0)$.

Def: The link of this singularity is

$$L = C \cap \partial B(\varepsilon) = \{(z,w) \mid (z,w) \in C, |z|^2 + |w|^2 = \varepsilon^2\}$$

for small ε.





<u>Note</u>: L is the intersection of a 2- and 3-dimensional space in $\mathbb{C}^2 = \mathbb{R}^4$ so it is a 1-dimensional real curve (or multicurve - called a <u>link</u>).

Write $z = re^{2\pi i\theta}$, $w = Re^{2\pi i\psi}$ w/ $r, R \geq 0$

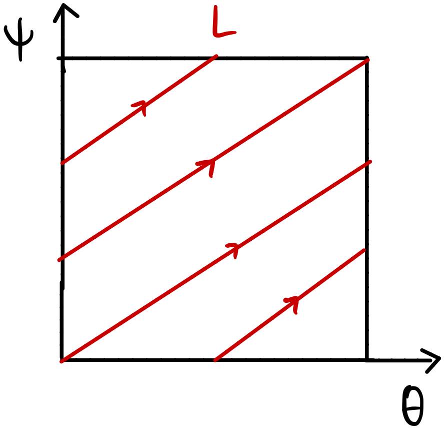
- $z^2 = w^3 \Rightarrow r^2 = R^3 \Rightarrow z = R^{3/2} e^{2\pi i\theta}$

  $\Rightarrow 2\theta = 3\psi \mod \mathbb{Z}$ so

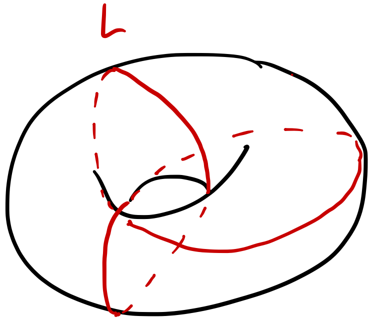
  $\psi = 2/3\,\theta + k/3$ for some $k \in \mathbb{Z}$

- $|z|^2 + |w|^2 = \varepsilon^2 \Rightarrow \underbrace{R^3 + R^2 = \varepsilon^2}$

$\exists!\ R > 0$ satisfying this.



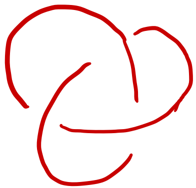
glue top to bottom + left to right



$2\theta = 3\psi \mod \mathbb{Z}$

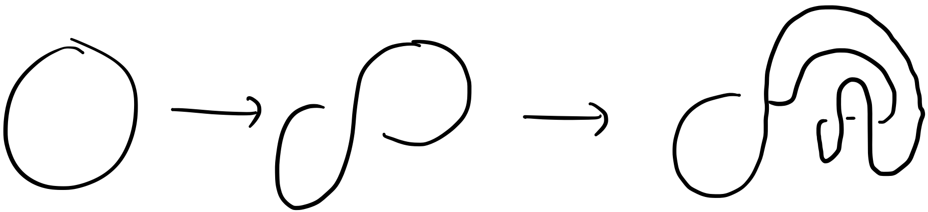
$\psi = 2/3\,\theta + k/3, \; k \in \mathbb{Z}$

$S^1 \times S^1 \subseteq S^3$

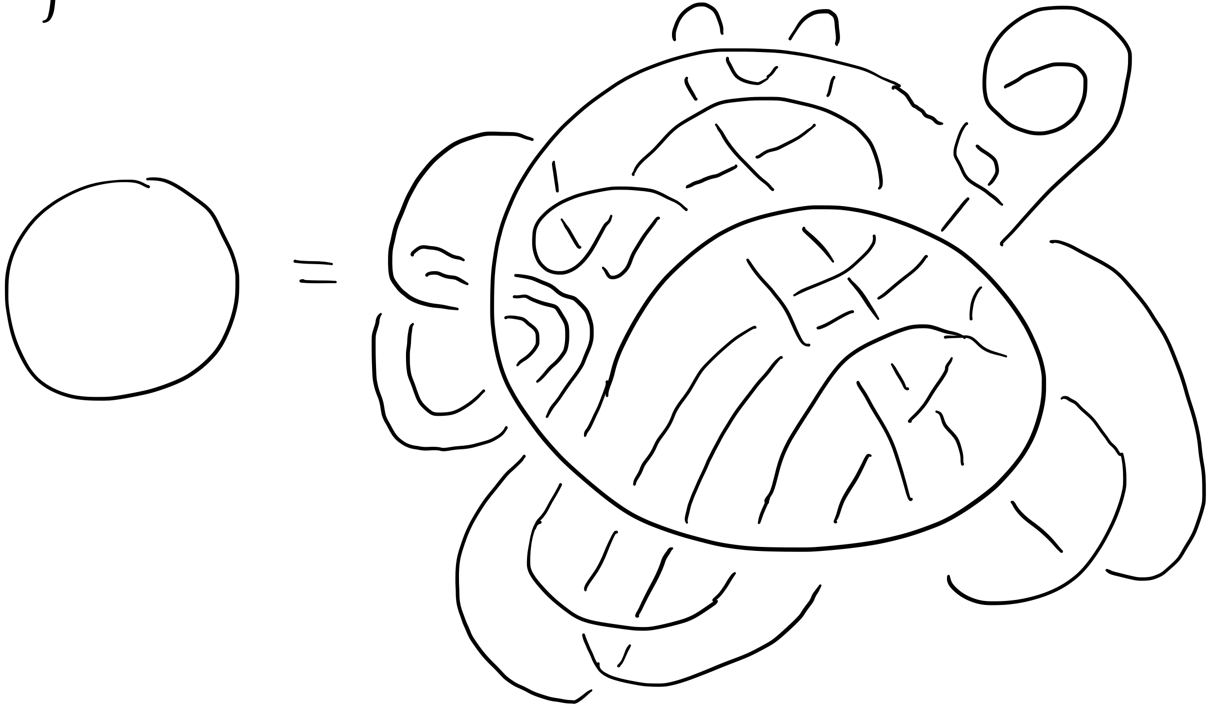
L = 

(2,3) torus knot (trefoil)

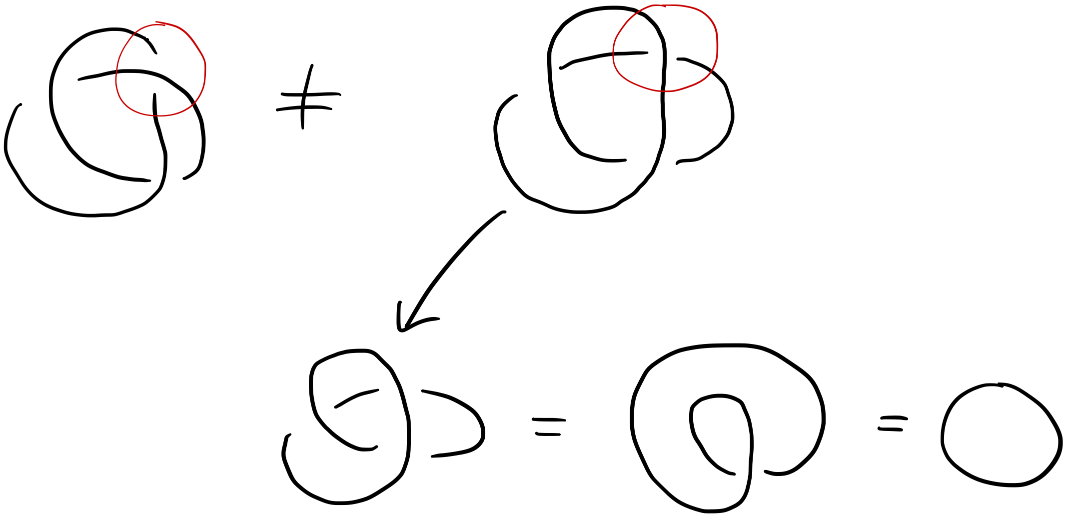
In knot theory, one typically studies knots up to isotopy: you can deform one knot to the other without passing the knot through itself.



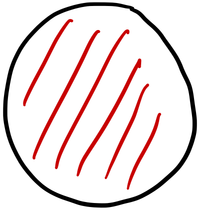
Two knots can be isotopic but "look" very different!

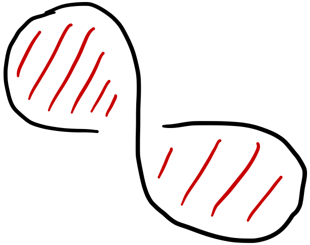


Not allowed to "change crossings"!



Note: The unknot is the unique knot that bounds a (smooth) disk in S^3 (or $\mathbb{R}^3$)

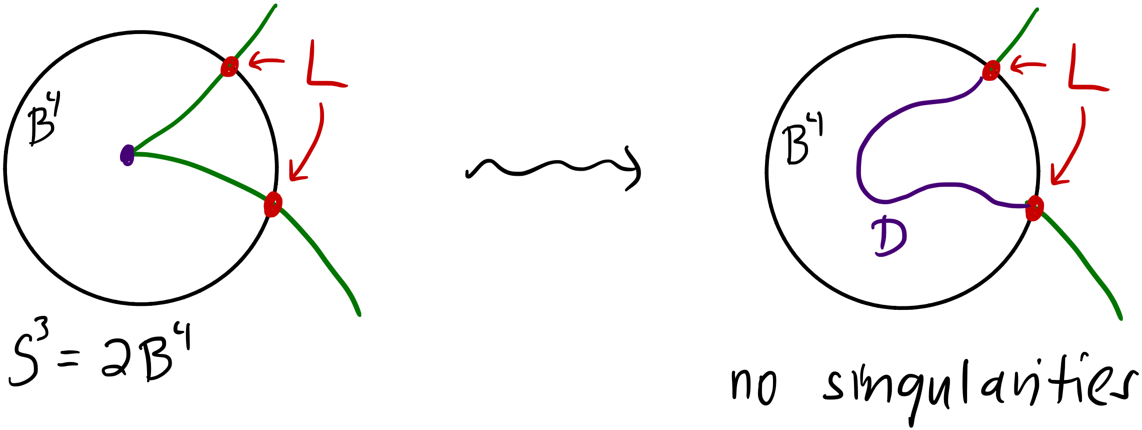




Q. What are the knots in S^3 (or $\mathbb{R}^3$) that bound a smooth disk in $B^4 = \{(z,w) \mid |z|^2 + |w|^2 \leq 1\}$ (or $\mathbb{R}^4_- = \{(x_1, \dots, x_4) \mid x_4 \leq 0\}$).

Such a knot is called (smoothly) slice.

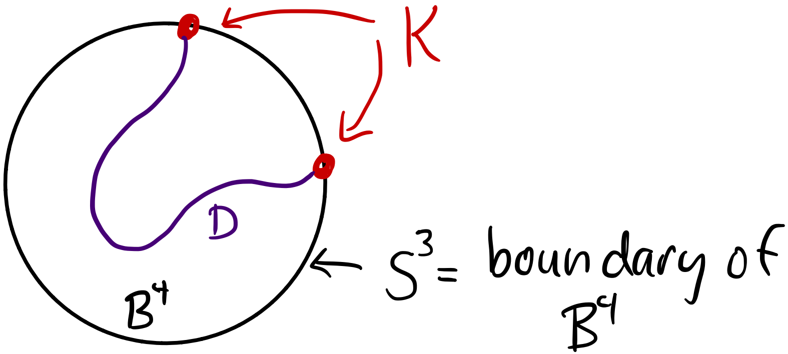
Fox and Milnor studied these in the 60's as a way to smooth singularities.



If L is slice, can replace singularity with smooth disk $D \subseteq B^4$.

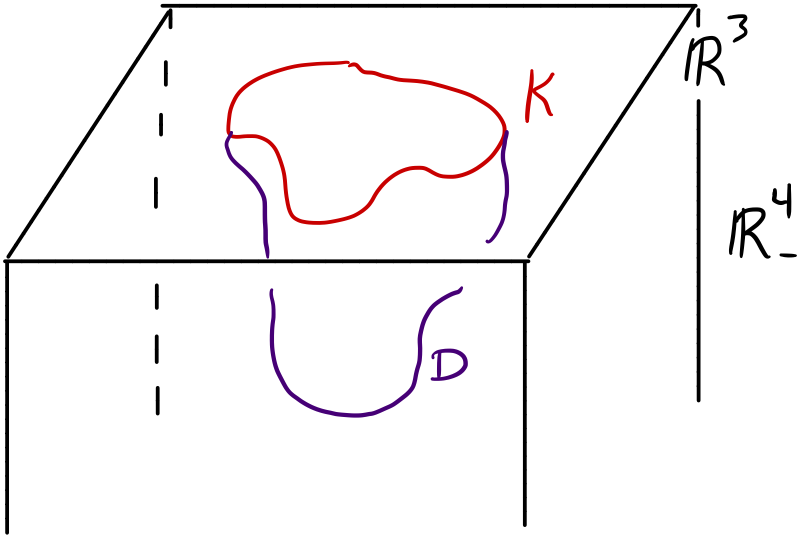
However, it turns out that the link of a singularity is never slice (except in the trivial case)!

Def: A knot $K \subseteq S^3 = \partial B^4$ is <u>slice</u> if $K = \partial D$ is the boundary of a smoothly embedded disk D in B^4.

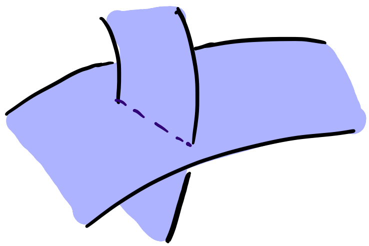


Equivalently:

A knot $K \subseteq \mathbb{R}^3 = \partial B^4_-$ is <u>slice</u> if $K = \partial D$ with D a smoothly embedded in $\mathbb{R}^4_-$.

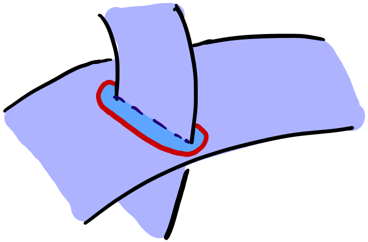


Ex: A knot is ribbon if it is the boundary of an immersed disk in $\mathbb{R}^3$ (or S^3) with "ribbon singularities":

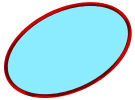


Observation: Every ribbon knot is slice.

Pf: Take a small disk around singularity and push it into $\mathbb{R}^4_+$ (or B^4)

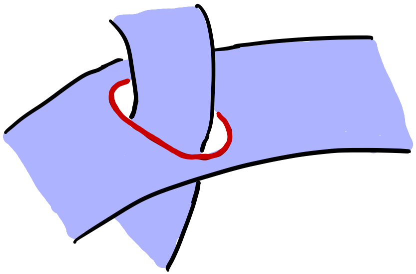


push interior of

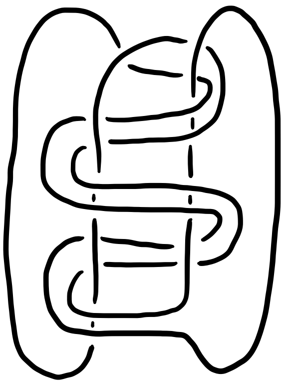


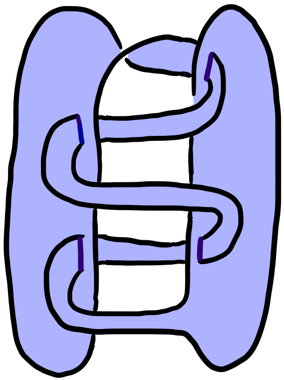
into interior of $\mathbb{R}^4_-$.

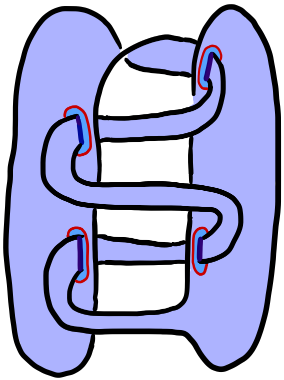
What is left in $\mathbb{R}^3$:



To get disk in $\mathbb{R}^4_-$, attach lower hemisphere of $S^2 = \{(x,y,0,v) \mid x^2+y^2+v^2=1\} \subseteq \mathbb{R}^4$ to red curve

$K = 8_9$ 

8_9 is ribbon 

Slice disk for 8_9 

$\Rightarrow$ 8_9 is slice but does not bound an embedded disk in $\mathbb{R}^3$!

Biggest open problem in knot concordance:

Slice-ribbon conjecture: Every (smoothly) slice knot is ribbon.

Note: This problem is extremely difficult since every ribbon knot has a slice disk that is not even is isotopic to any ribbon disk!

Thus, cannot start with a slice disk and deform it to become a ribbon disk.

[For the experts]

<u>Ex:</u> Let S be a smoothly embedded non-trivial 2-knot, $S^2 \hookrightarrow S^4$. Let U = unknot, and D = standard disk with $\partial D = U$. Push U into B^4 and then take a connected sum with S. Then $U = \partial \mathring{S}$ (S punctured) and $\pi_1(B^4 \setminus S^\circ) = \pi_1(S^4 \setminus S)$ is non-abelian since S is non-trivial.

<u>Fact</u>: If D is a ribbon disk for K then

$$\pi_1(S^3 \setminus K) \xrightarrow{i_*} \mathrel{\mkern-14mu}\rightarrow \pi_1(B^4 \setminus \underset{\substack{\uparrow \\ \text{pushed in}}}{D})$$

is surjective.

In example:

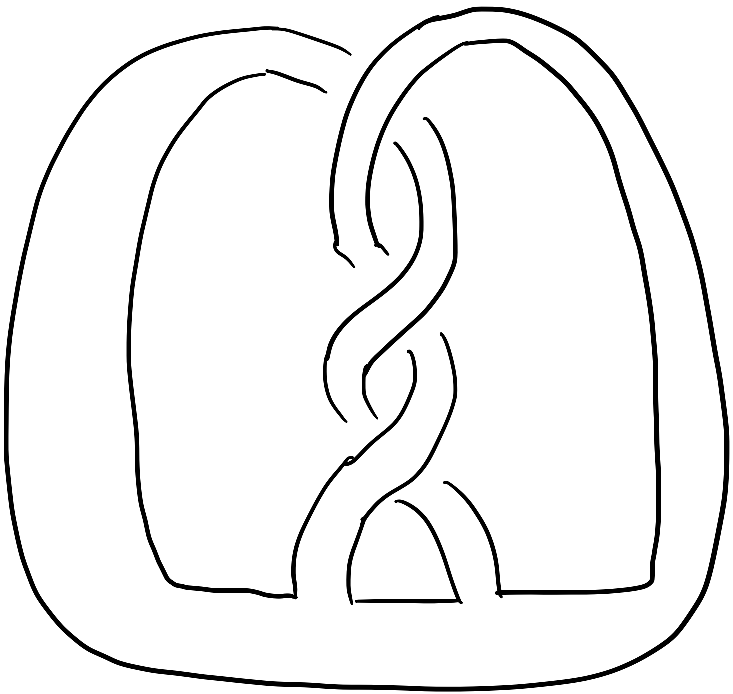
$$\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}}}{\pi_1(S^3 \setminus \text{unknot})} \longrightarrow \underset{\substack{\uparrow \\ \text{non-abelian}}}{\pi_1(B^4 \setminus \mathring{S})}$$

$\Rightarrow$ $\mathring{S}$ is a slice disk that is not isotopic to <u>any</u> ribbon disk.

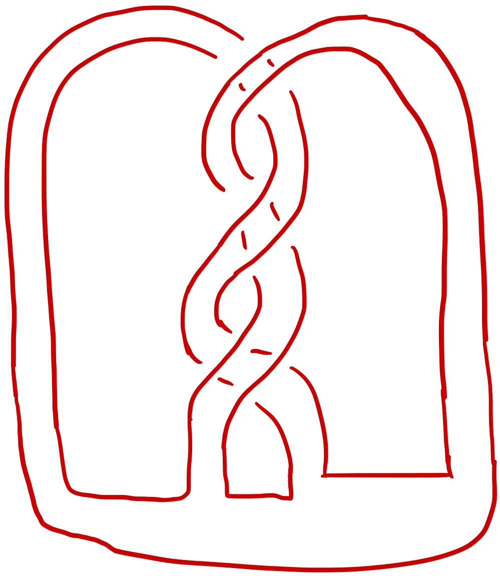
Another example (using "movie moves")

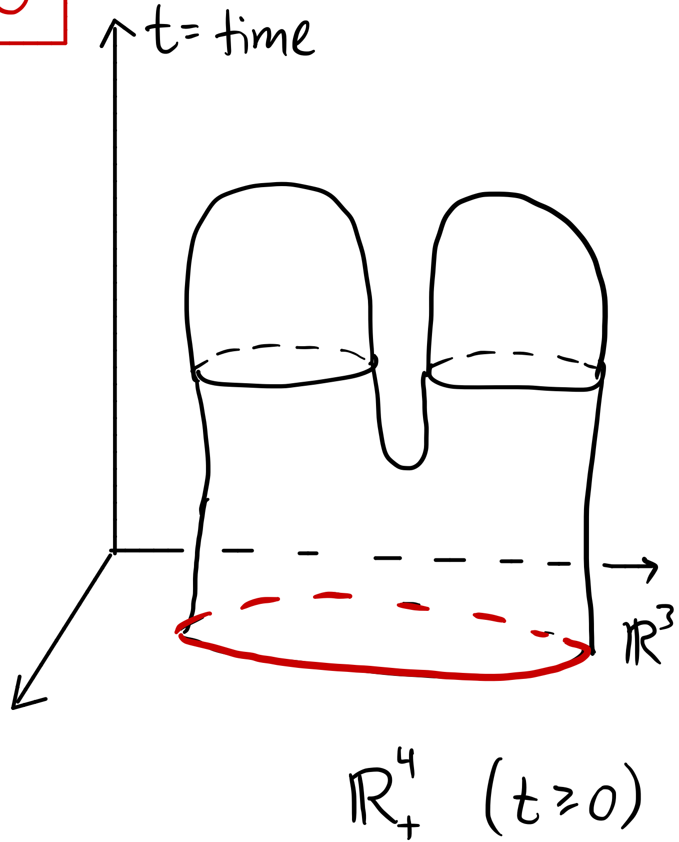
We can look at level set of a disk in $\mathbb{R}^4_+$.

9_{46} is slice

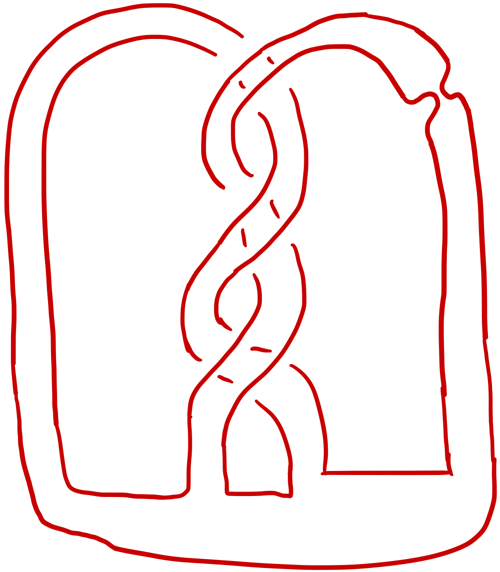


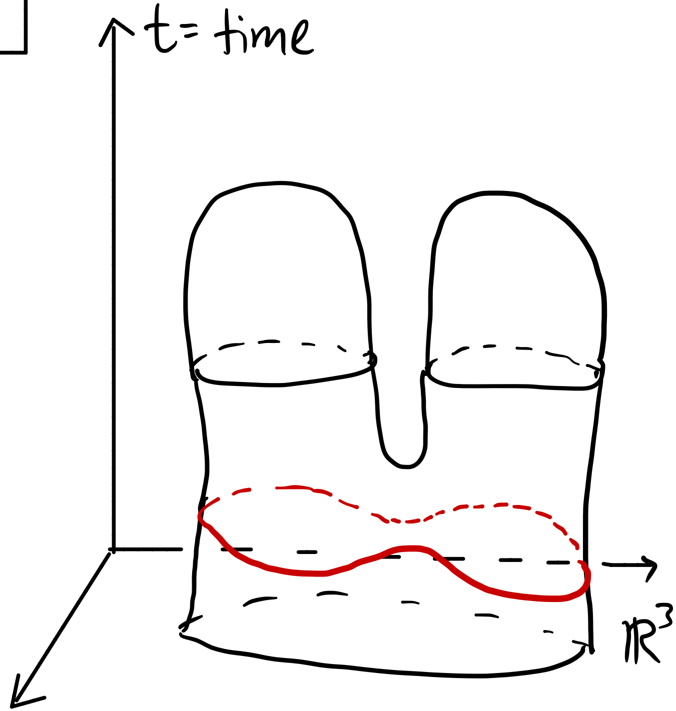
$\boxed{t=0}$





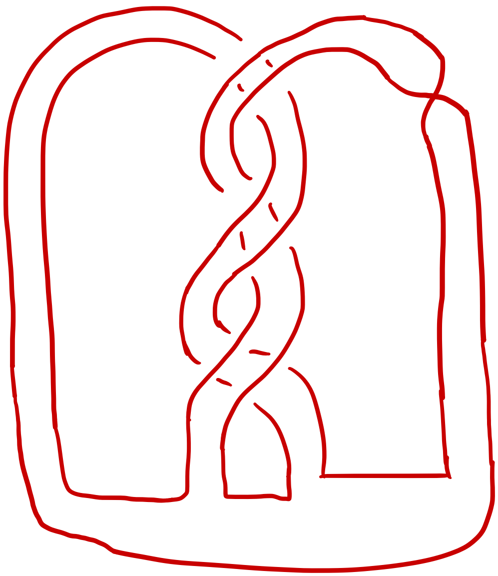
$\boxed{t = 1/8}$

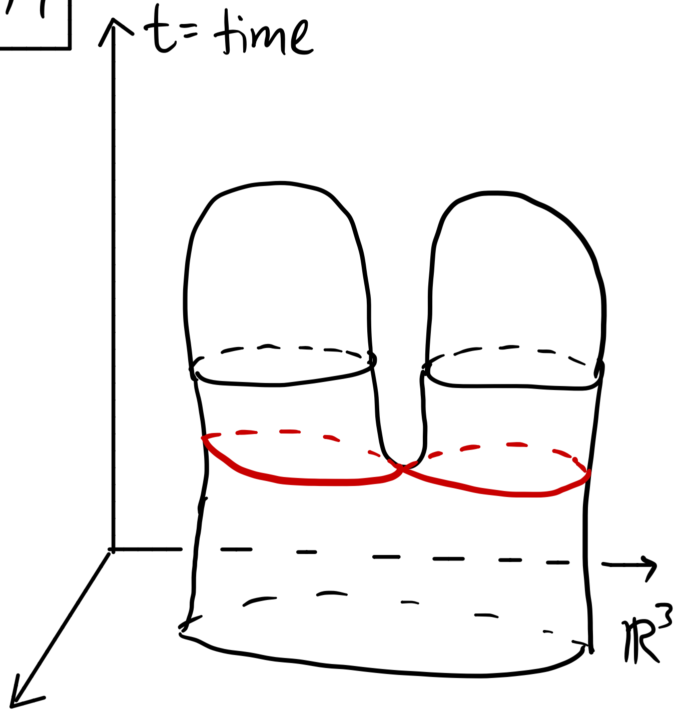




$\mathbb{R}^4_+$ $(t \geq 0)$

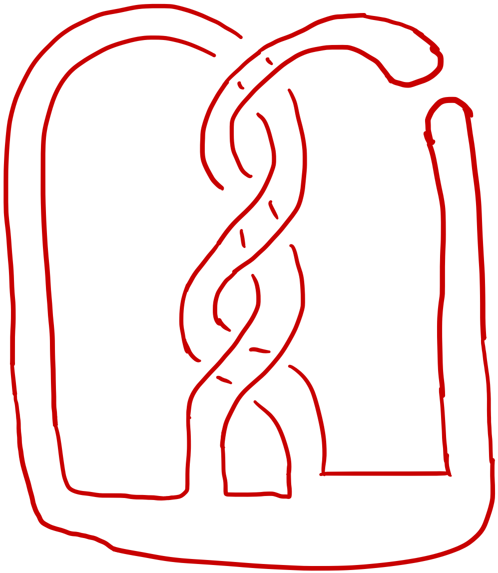
$\boxed{t = 1/4}$

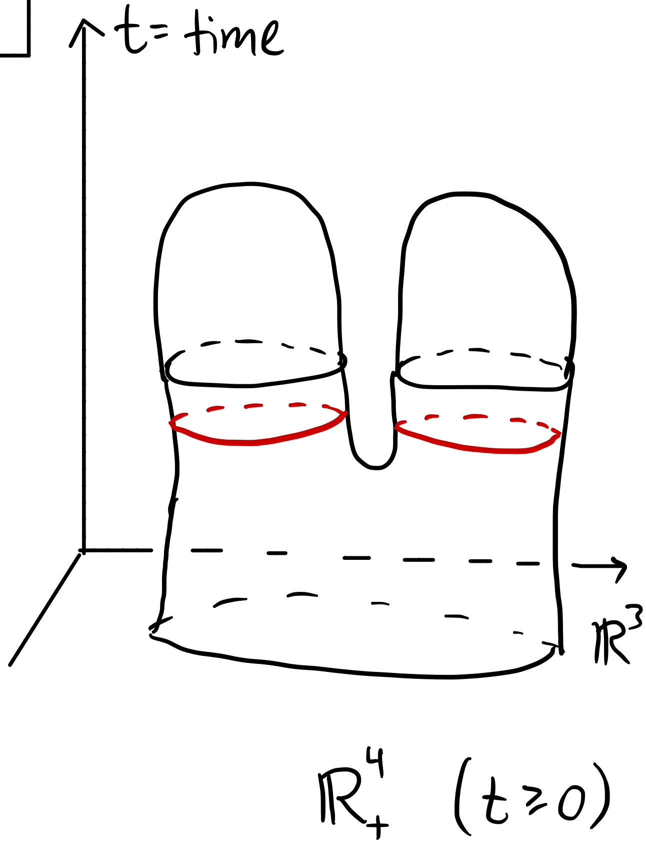




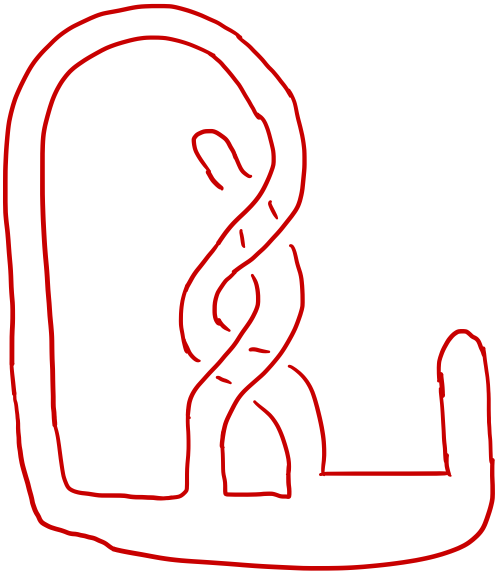
$\mathbb{R}^4_+ \quad (t \geq 0)$

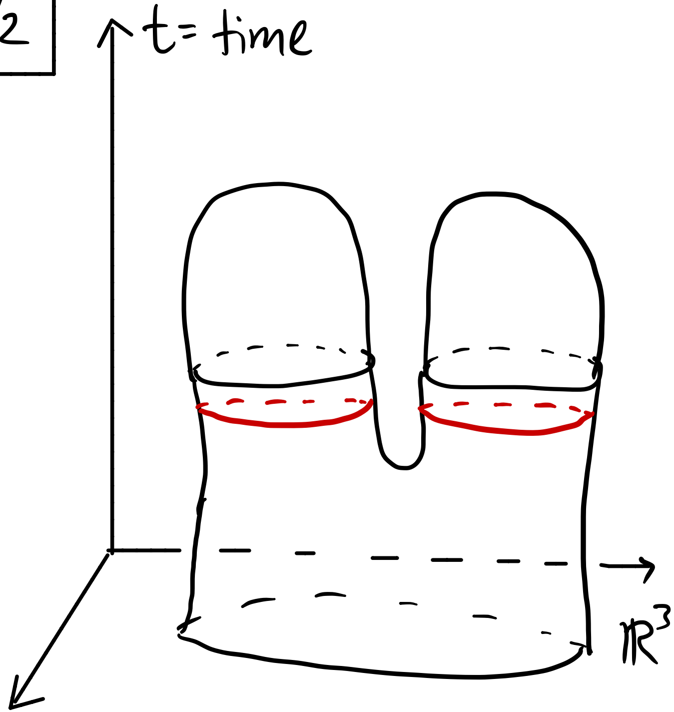
$t = 3/8$





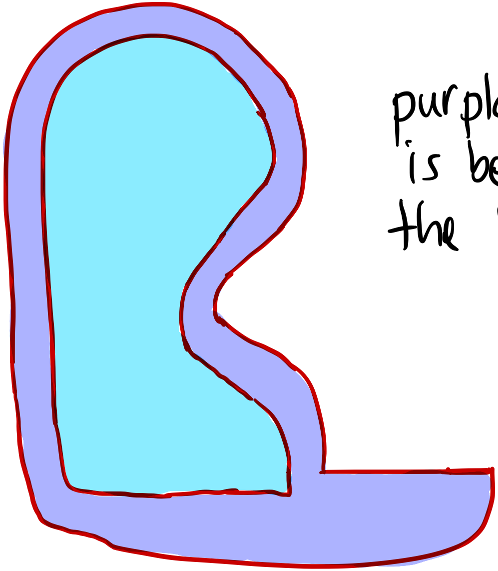
$\boxed{t = 1/2}$



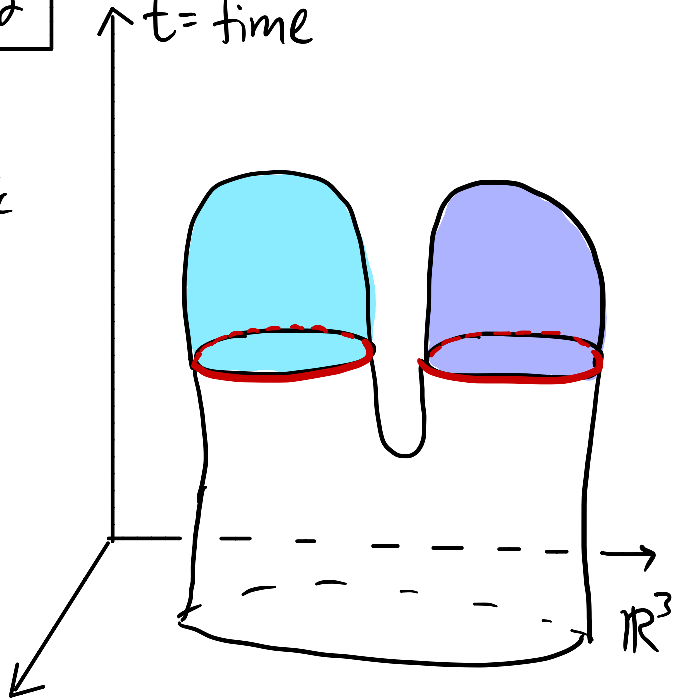


$\mathbb{R}^4_+ \quad (t \geq 0)$

$\boxed{t \geq 1/2}$



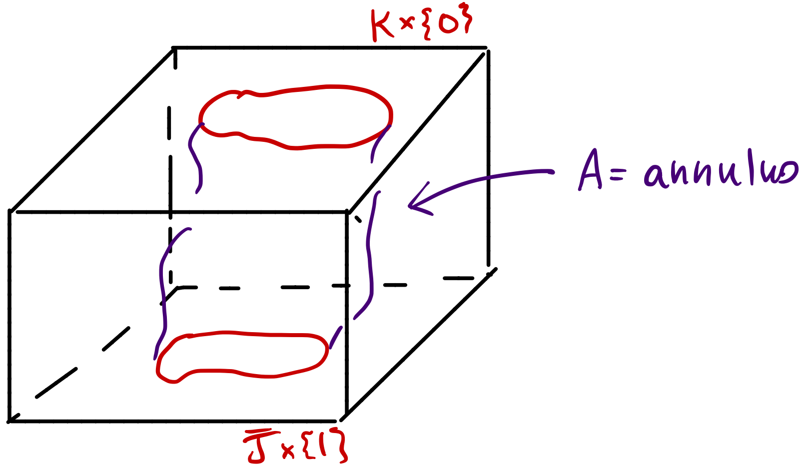
purple disk is behind the blue



$\mathbb{R}^4_+ \ (t \geq 0)$

We can put an 4-dimensional equivalence relation on knots.

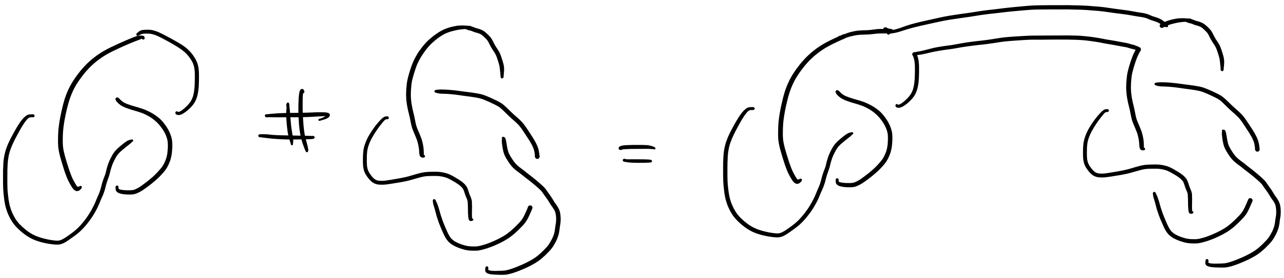
Def: Let K and J be knots in $\mathbb{R}^3$. We say that K is concordant to J if $K \times \{0\}$ and $\bar{J} \times \{1\}$ cobound a smoothly embedded annulus in $\mathbb{R}^3 \times [0,1]$.



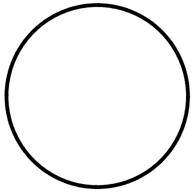
Concordance group

Let $\mathcal{C} = \{\text{knots}\}/\sim$ $\quad K \sim J$ if they are concordant.

Then $\mathcal{C}$ is a group under connected sum.

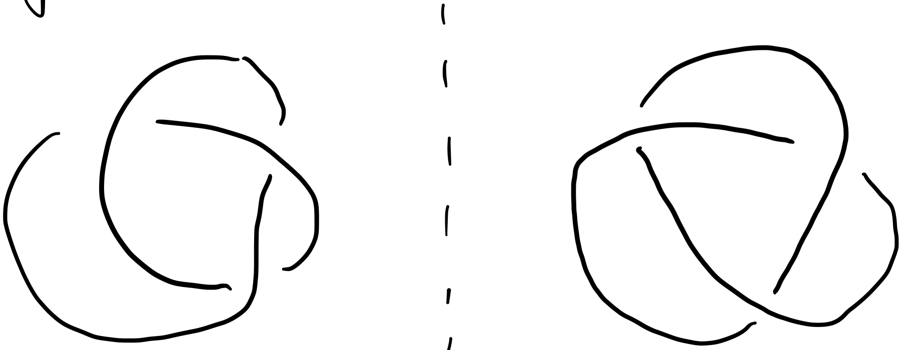


* need oriented knots.

 = identity

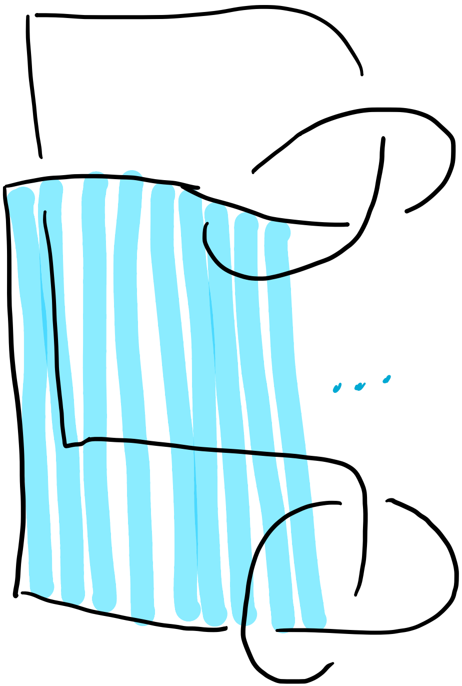
Inverse of K is $\bar{K}$.

For any K, $K \# \bar{K}$ is slice where

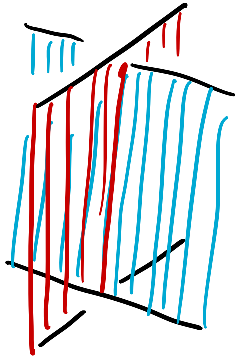


K $\qquad$ $\bar{K}$ = mirror image

Pf that $K \# \bar{K}$ is slice (ribbon)



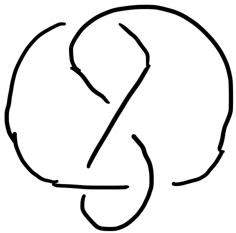
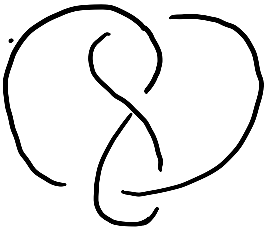
Make immersed disk by lines from K to $\bar{K}$.
The only self-intersection are ribbon singularities



$\mathcal{C}$ is a non finitely generated abelian group.

We don't know what $\mathcal{C}$ is.

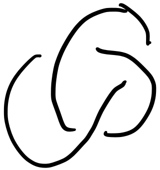
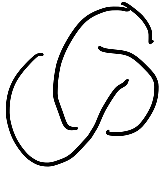
- $\mathcal{C}$ contains elements that are 2-torsion.

 $=$ 

4_1 $\qquad\qquad$ $\overline{4}_1$

$\Rightarrow 2\,4_1 = 0$ and 4_1 is not slice ($4_1 \neq 0$)

- $\mathcal{C}$ contains elements of infinite order

 $\#$ $\cdots$ $\#$  is never slice.

<u>Thm</u> (Levine '60's) $\exists$ surjective homomorphism

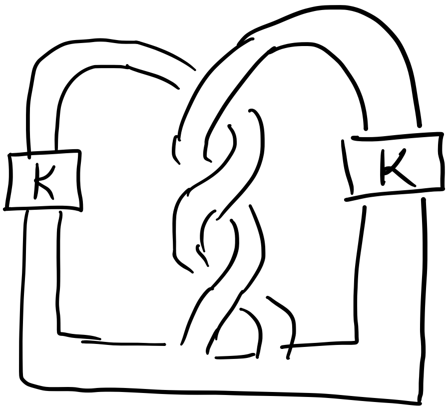
$$\mathcal{C} \xrightarrow{\pi} \mathcal{A} \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}_4^{\infty}$$

$\uparrow$
algebraic concordance group
(Witt group of Seifert matrices)

Q. Are all torsion elements, 2-torsion?

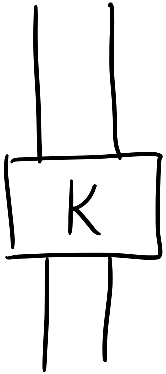
- $\ker(\pi)$ is non-trivial (in higher dimensions π is an $\cong$)

<u>Thm</u> (Casson-Gordon, Gilmer): $\ker \pi \neq 0$.



K = trefoil

Tie strand into K



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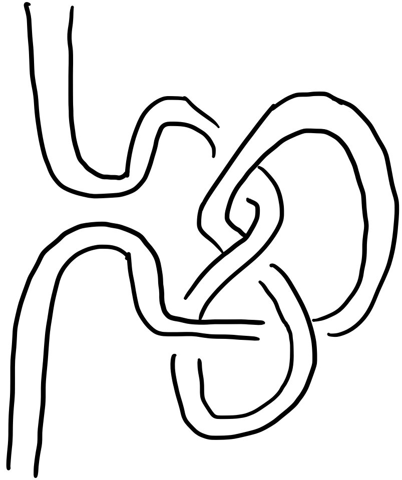
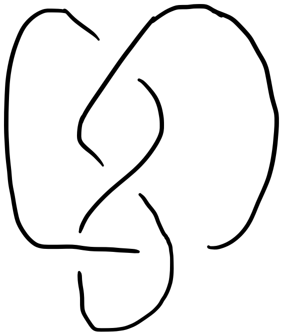


K =



n-solvable filtration

Cochran-Orr-Teichner defined filtration

$$\cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

$K \in \mathcal{F}_0 \iff \mathrm{Arf}(K) = 0$ Arf invariant

$K \in \mathcal{F}_{0.5} \iff K \in \ker(\pi)$ Algebraically slice

$K \in \mathcal{F}_{1.5} \implies$ Casson-Gordon invariants vanish.

Thm (Cochran-H-Leidy): For each $n \geq 0$,

$\mathcal{F}_n / \mathcal{F}_{n.5}$ contains $\displaystyle\bigoplus_{\substack{p(t) \\ \text{symmetric} \\ \text{irreducible}}} \left(\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \right)$

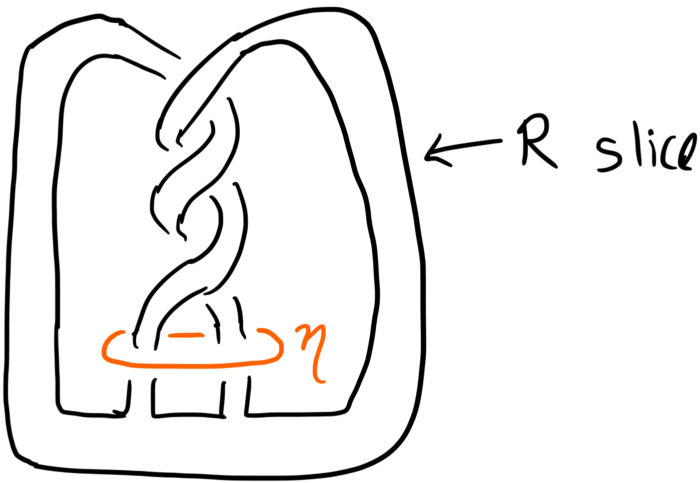
$n=0$: Milnor-Tristram, Levine (60's)

$n=1$: Jiang, Livingston (80's)

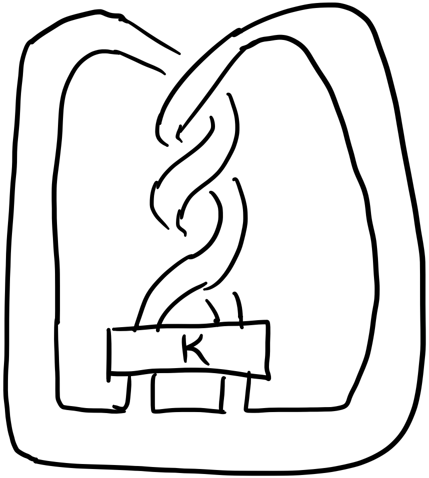
$n=2$: Cochran-Teichner ('02)

Operators on $\mathcal{C}$

Def: A pattern P is a slice knot R and unknot η disjoint from R, such that η bounds a surface disjoint from R.

$P =$ 

$P : \mathcal{C} \longrightarrow \mathcal{C}$ (not a homomorphism)

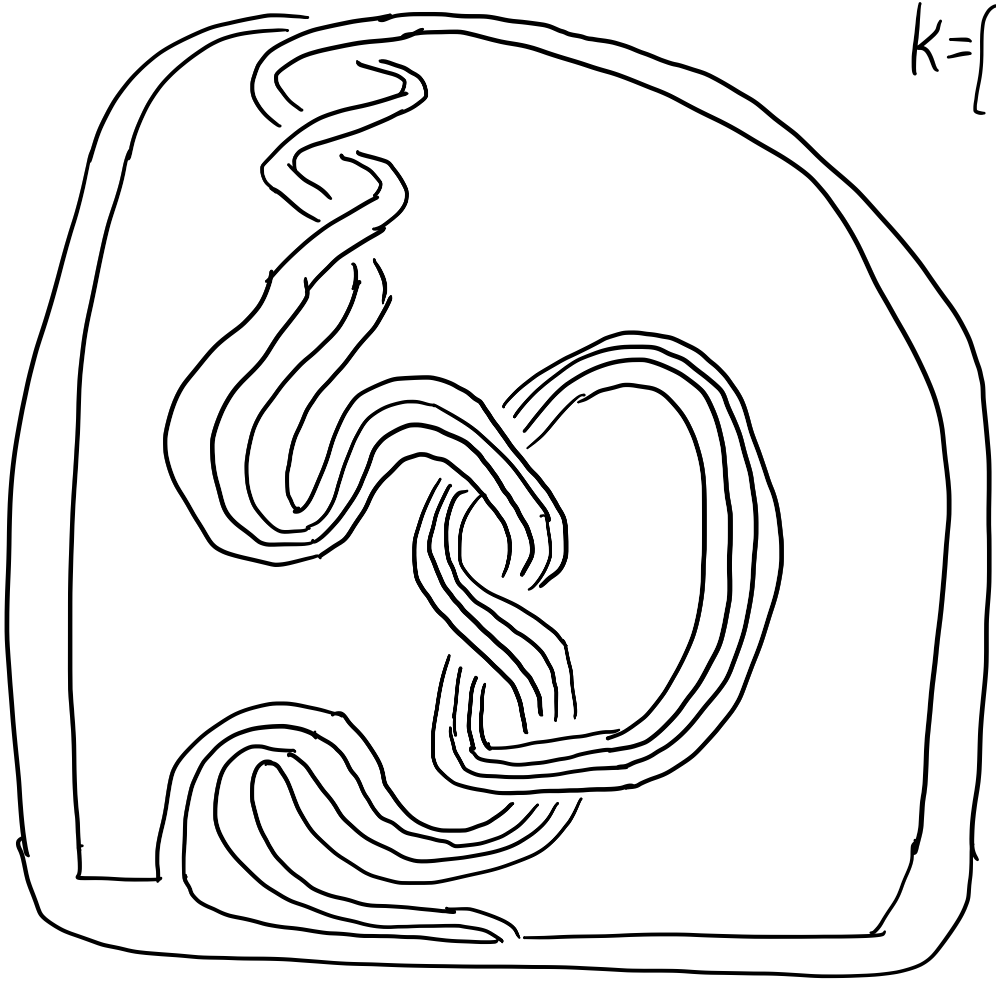
$P(K) =$ 

tie strands going through η into K

satellite operator.

Ex:

$K =$ 

$P(K) =$ 

$$P : \mathcal{F}_n \longrightarrow \mathcal{F}_{n+1}.$$

Hence $P^n(K) = \underbrace{P(P(\cdots P}_{n \text{ times}}(K)) \in \mathcal{F}_n$

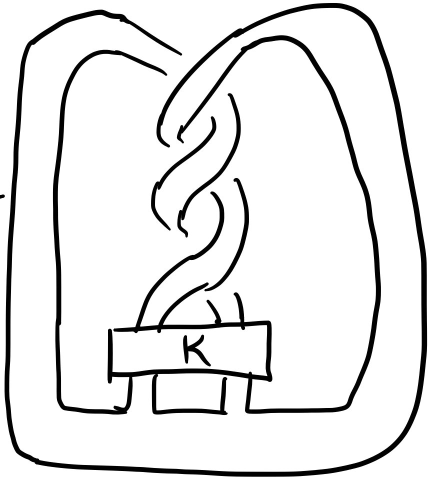
for any K with Arf invariant zero.

Exs of $\mathbb{Z}^\infty$ and $\mathbb{Z}_2^\infty \in \mathcal{F}_n / \mathcal{F}_{n.5}$ are

constructed this way!

Q. When is P injective?

<u>Conjecture:</u> Q is injective

$Q(K)=$ 

$Q(K)$ is slice $\Leftrightarrow$ K is slice

Every such P would re-embed $\mathcal{C}$ into itself.

Known: There is a subgroup of $\mathcal{C}$ on which P^n is injective for $\forall n$.

Satellite operators give a way to construct elements in $\mathcal{F}_n$. The difficult part is to show $P^n(K)$ is not slice (or even in $\mathcal{F}_{n.5}$)!!!

- Use invariants of knots such as L^2-signatures, d-invts and τ invariants from Heegaard Floer homology, etc.

Note: There is no known algorithm to determine if a knot is slice!!!

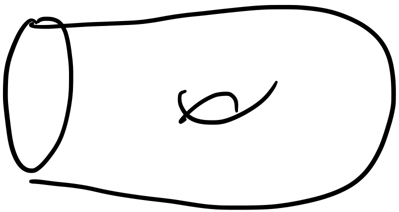
Q. Is the Conway knot slice?

known to topologically slice but unknown if it is smoothly slice!

Would like some notion of distance where mage(P^n) is getting smaller as $n \to \infty$.

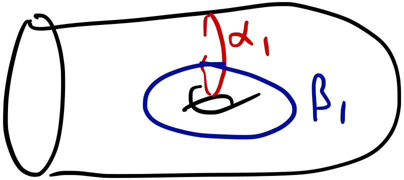
Symmetric gropes

<u>Def</u>: A <u>grope</u> of height 1 is a compact oriented surface G_1 with $|\partial \Sigma| = 1$.



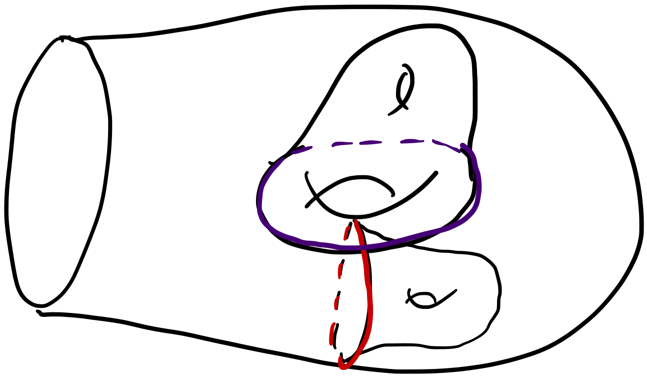
G_1

Let $\{\alpha_1, \dots, \alpha_{2g}\}$ be a standard symplectic basis of curves for $H_1(G_1)$ on G_1, $g = \text{genus}(G_1)$



A grope of height n+1 is obtained by attaching gropes of height n to $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$.

height 2 grope

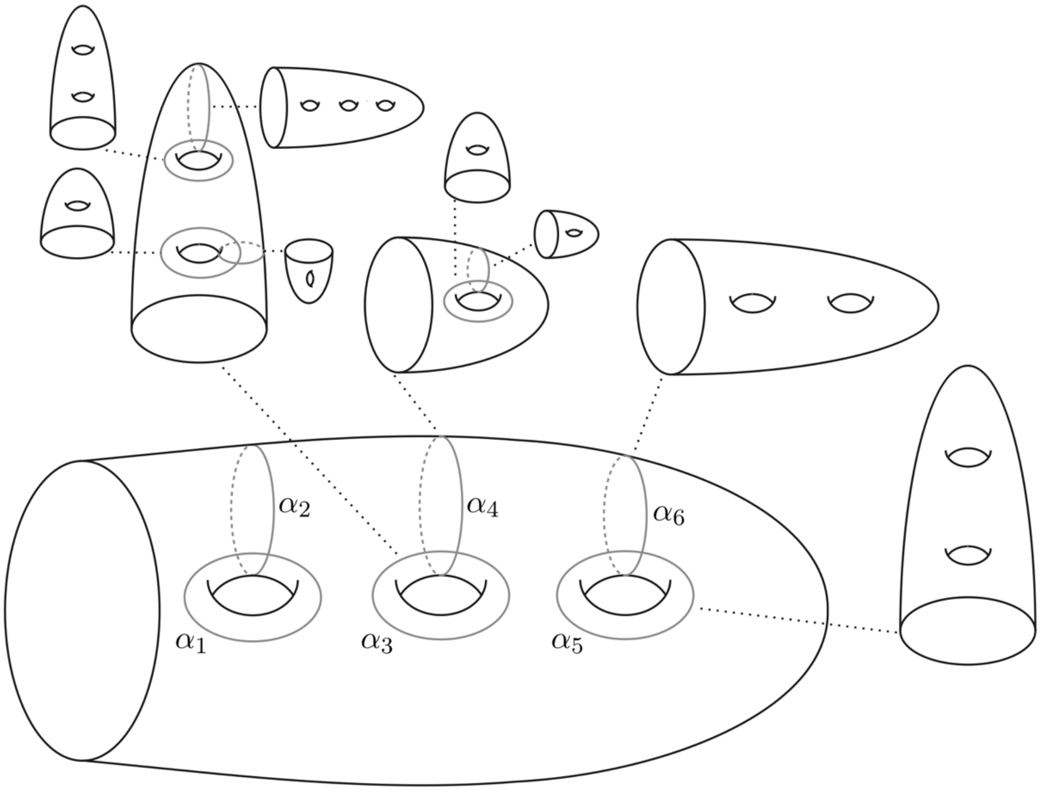


Def: A **branched symmetric grope** is defined as follows:

Let $\Sigma_{1:1}$ be a compact connected orientable surface of genus g, with a standard sympl. basis of curves $\{\alpha_1, \dots, \alpha_{2g}\}$ with α_{2i-1} dual to α_{2i}. Attach to each α_i, a grope of height m_i s.t. $m_{2i-1} = m_{2i}$, no subsurface of which is a disk.

Let $n_i = m_{2i}$.

$n_1 = m_1 = m_2 = 0$ $n_2 = m_3 = m_4 = 2$ $n_3 = m_5 = m_6 = 1$



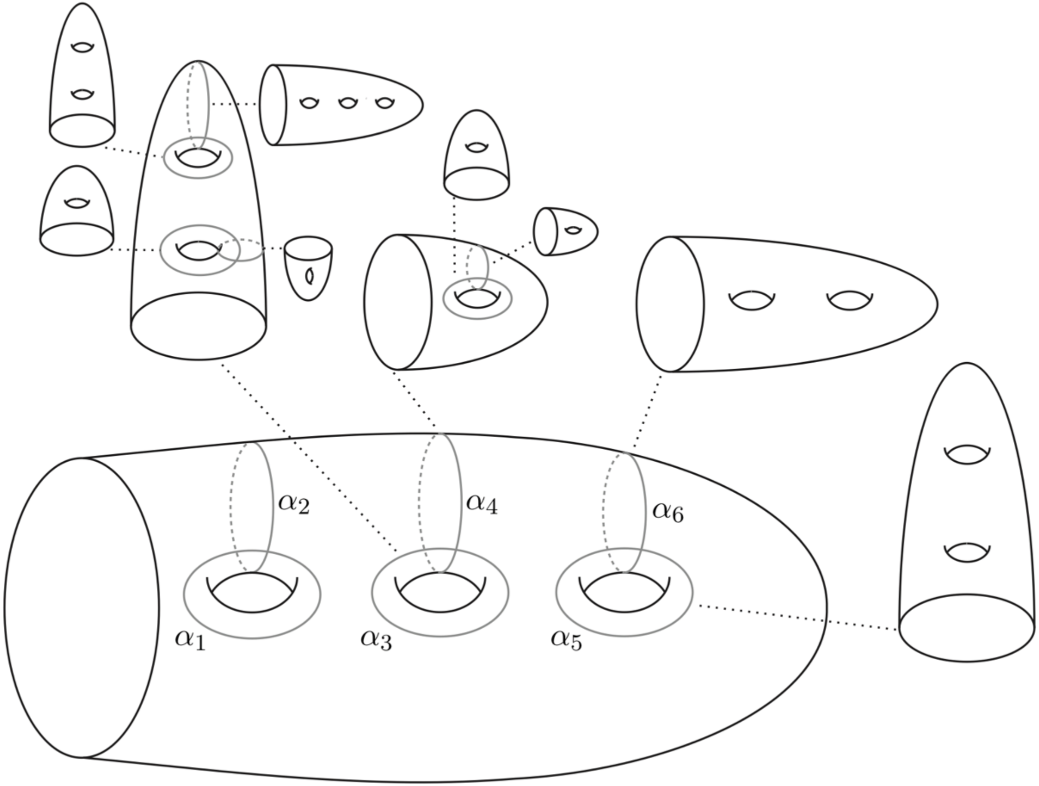
Let Σ be a branched symmetric grope.

Define $g_1 = \text{genus}(\Sigma_1)$

g_2^i = sum of genera of first stage surfaces attached to $\alpha_{2i-1}, \alpha_{2i}$.

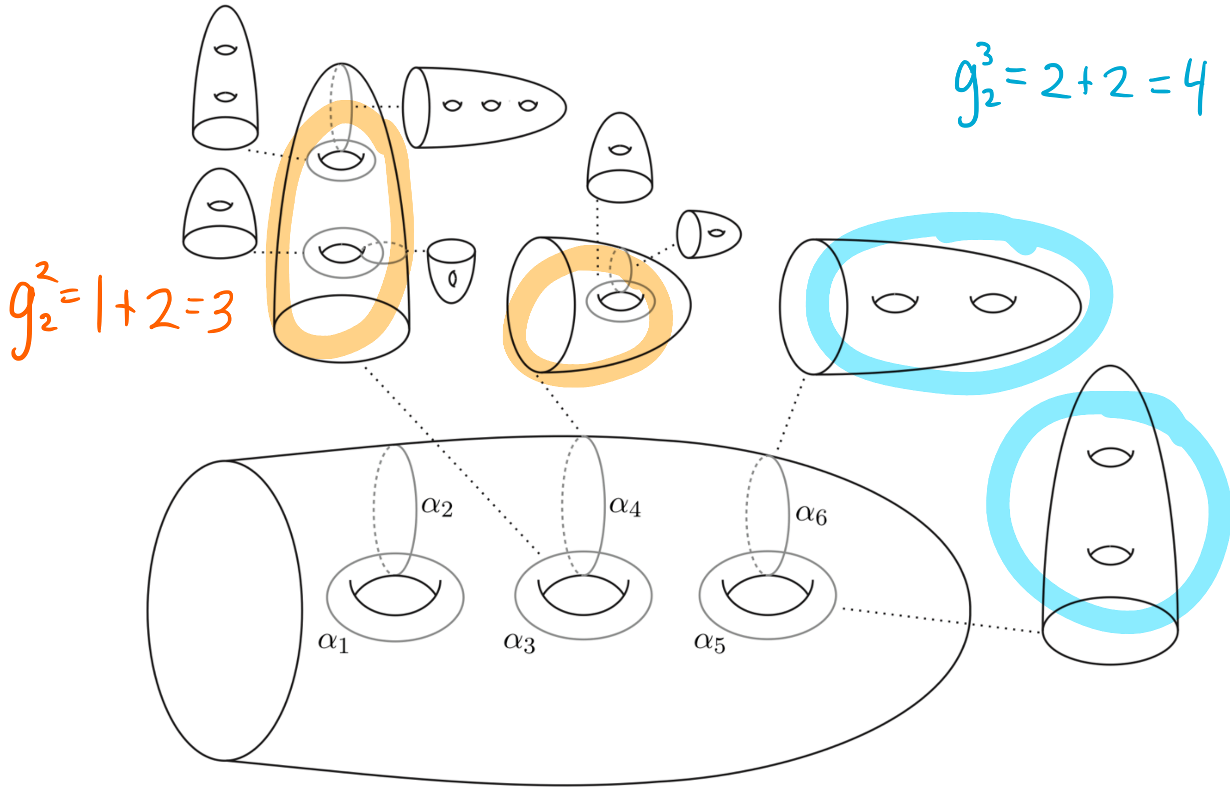
$\vdots$

$g_{n_i+1}^i$ = sum of genera of n_i stage surfaces attached to $\alpha_{2i-1}, \alpha_{2i}$.

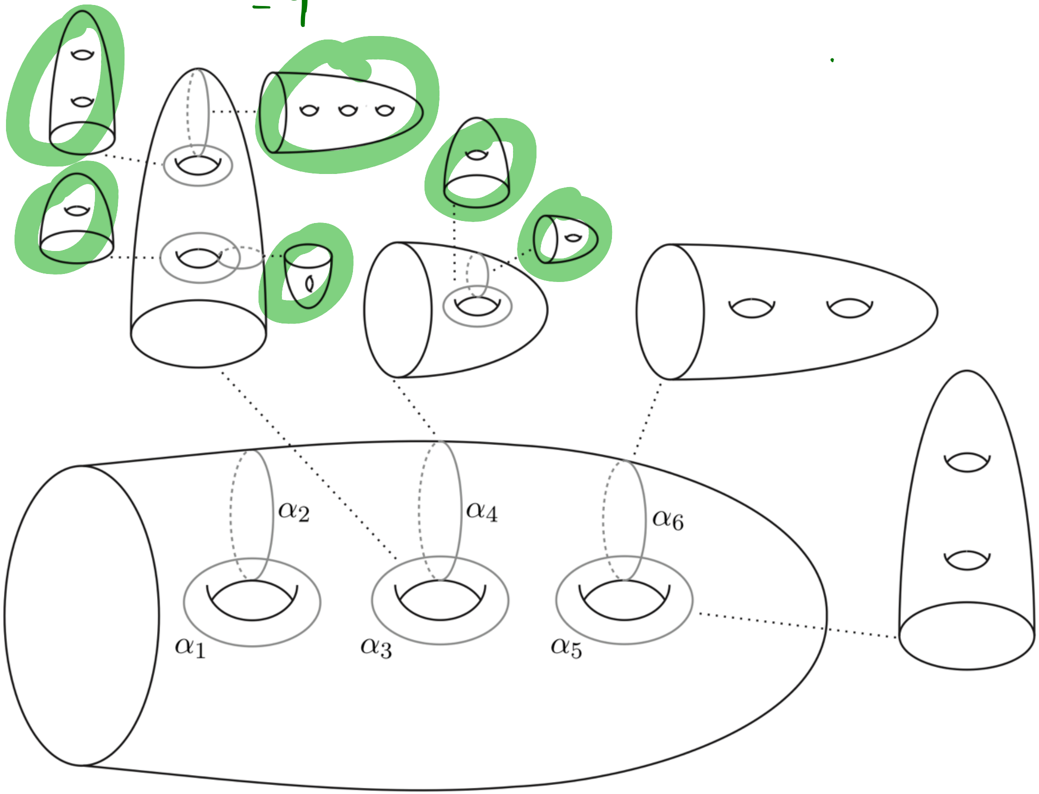


$g_1 = 3$

No g_2^1 since $n_1 = m_1 = m_2$.



$g_3^2 = 2+1+3+1+1+1$
$= 9$



<u>Note:</u> For each $1 \le i \le g$, and $2 \le k \le n_i + 1$,

$$g_k^i \geq 2 g_{k-1}^i$$

$$\implies \boxed{g_k^i \geq 2^{k-1}}$$

Let $q \geq 1$ be a real number and Σ a branched symmetric grope. Define

$$\|\Sigma\|_q := \sum_{i=1}^{g_1} \frac{1}{q^{n_i}} \left(1 - \sum_{k=2}^{n_i+1} \frac{1}{g_k^i}\right)$$

<u>Def</u>: If K, J are knots, define

$$d^q(K,J) := \inf\left\{ \|\Sigma\|_q \;\middle|\; \begin{array}{c} \Sigma \text{ is a branched symmetric grope} \\ \text{embedded in } S^3 \times I \text{ with boundary} \\ K \times \{0\} \text{ and } \bar{J} \times \{1\} \end{array} \right\}$$

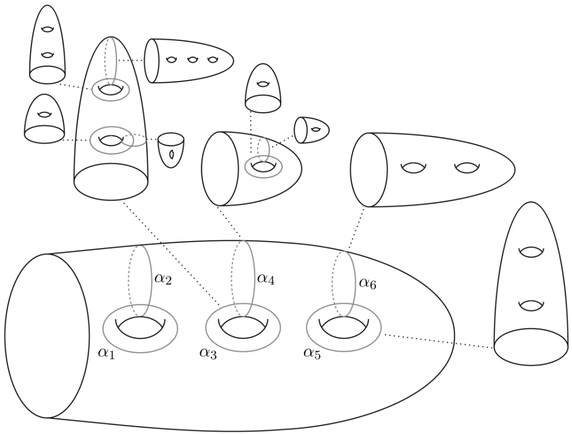
<u>Note</u>: Any two knot cobound a surface.

Ex: If K has bounds a genus 1 surface Σ and $\mathrm{Arf}(K) \neq 0$ then K cannot bound a (symmetric) height 2 grope. So

$$d^q(K, \text{unknot}) = g(\Sigma) = 1.$$

Ex: $\frac{1}{2q} \leq d^q($  ,  $) \leq \frac{27}{16q}$

$\Sigma =$



$$\|\Sigma\|_q = \left(\frac{1}{q^0}\cdot 1\right) + \frac{1}{q^2}\left(1-\frac{1}{3}-\frac{1}{9}\right) + \frac{1}{q^1}\left(1-\frac{1}{4}\right) = 1 + \frac{5}{9q^2} + \frac{3}{4q}$$

$i=1$ $\quad$ $i=2$ $\quad$ $i=3$

__Note:__ The only way one could get zero is to have a (symmetric) grope of arbitrarily long height with all genus 1 surfaces at each stage, or an annulus.

$$\|\Sigma\|_q = \frac{1}{q^{n_1}}\left(1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^{n_1}}\right) \longrightarrow 0 \quad \text{as } n_1 \to \infty$$

Prop (Cochran-H-Powell): For $q \geq 1$, the function d^q determines a pseudo-metric on $\mathcal{C}$.

- Need to show $\|\Sigma\|_q \geq 0$ for any Σ.

Prop: If K does not bound a grope of height n then

$$d(K, \text{unknot}) \geq \frac{1}{(2q)^{n-2}}.$$

Thm (Cochran-Orr-Teichner): If K bounds a height n grope then $K \in \mathcal{F}_{n-2}$.

Prop (Cochran-H-Powell): If P is a pattern then $P: \mathcal{C} \longrightarrow \mathcal{C}$ is a contraction w.r.t. d^q for $q > gw(P) := \#$ of times R goes through η.

<u>Thm</u> (Cochran-H-Powell): For any $q>1$ there exists uncountably many sequences of knots $\{K_i\}$ s.t.

$$d^q(K_i, \text{unknot}) > 0 \quad \forall\, i \quad \text{but}$$
$$d^q(K_i, \text{unknot}) \longrightarrow 0 \quad \text{as} \quad i \to \infty.$$

Hence the topology on $(\mathcal{C}, d^q)$ is not discrete for $q>1$.