Homology Equivalence of Groups and Spaces.

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Outline

1. Define $\rho_n^{(2)} : \{3\text{-mflds}\} \longrightarrow \mathbb{R}$ for each $n \geq 0$.

2. <u>Thm</u> (H): $\rho_n^{(2)}$ is an invariant of homology cobordism.

3. <u>Thm</u> (H): The $\rho_n^{(2)}$ are independent. For each $n \geq 0$, image $\rho_n^{(2)}$ is dense in $\mathbb{R}$ and has infinite rank.

4. Define $\rho_n^{(2)}$ on concordance classes of links or string links by

   - link $L \longmapsto$ 0-surgery on $L \longmapsto \rho_n$(0-surge

   - string link $S \longmapsto$ closure of S ($\hat{S}$) $\longmapsto$ 0-surgery of $\hat{S}$ $\longmapsto \rho_n$(0-surgery of $\hat{S}$)

5\. Let $C(m) = \left\{ \begin{array}{l} \text{concordance group} \\ \text{of } m\text{-component string} \\ \text{links in } S^3 \end{array} \right\}$

and

$$\cdots \subset \mathcal{F}_{(n.5)}(m) \subset \mathcal{F}_{(n)}(m) \subset \cdots \subset \mathcal{F}_{(0.5)}(m) \subset \mathcal{F}_{(0)}(m) \subset C(m)$$

be filtration of $C(m)$ by (n)-solvable string links (as defined by Cochran-Orr-Teichner) for $n \in \mathbb{Z}^+/2$.

Thm(H): For each $m \geq 2, n \geq 1$,

$$\mathcal{F}_{(n)}(m) / \mathcal{F}_{(n.5)}(m)$$

is infinitely generated.

Remarks

1. Recently S. Chang + S. Weinberger show similar results for $4k-1 \geq 7$ mflds. They consider the $\rho^{(2)}$-invt of the universal cover $\tilde{M}$ of a $(4k-1 \geq 7)$-dim mfld M and show that if $\pi_1(M^{4k-1})$ has torsion then there exists an ∞ # of M_i homotopy equivalent but not homeomorphic to M since the $\rho^{(2)}(\tilde{M}_i)$ are distinct!

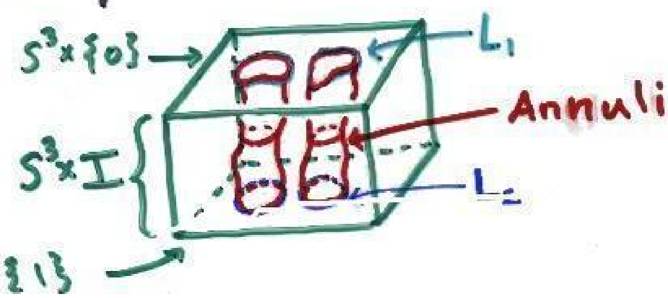
2. For knots, Cochran-Orr-Teichner show that for all n $\mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$ is non-zero. They also show that $\mathcal{F}_{(2)}/\mathcal{F}_{(2.5)}$ is infinitely generated. However, it is <u>unknown</u> whether $\mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$ is infinitely generated for <u>$n \geq 3$</u>.

M^3 is an oriented, closed, 3-manifold

<u>Def$^{\underline{n}}$</u> M_1^3 is <u>homology</u> <u>cobordant</u> to M_2^3, <u>$M_1 \sim_H M_2$</u>, if there exists smooth 4-mfld W such that $\partial W = M_1 \cup -M_2$ and $i_j : M_j \to W$ induce $\cong$ on $H_*(-;\mathbb{Z})$.

$$\mathcal{H}^3 := \{M^3\}/\sim_H$$

<u>Example</u>: $L_1, L_2 \hookrightarrow S^3$ links in S^3 are <u>concordant</u> if they cobound smoothly embedded annuli in $S^3 \times I$.



If L_1 concordant to L_2 then

$$M_{L_1} \sim_H M_{L_2} \;!$$

'' (0-surgery on L_1)

Homology Equivalence and Fundamental Gp.

A homology cobordism gives maps

$$i_j : M_j \longrightarrow W$$

which induce $\cong$ on H_* (i.e. i_j is homology equivalence)

Q. If $f: X \longrightarrow Y$ is a homology equivalence what is preserved under

$$f_*: \pi_1(X) \longrightarrow \pi_1(Y) \ ?$$

Example 1:
$$f_*: \underset{\substack{\| \\ H_1(X)}}{\frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}} \xrightarrow{\ \cong\ } \underset{\substack{\| \\ H_1(Y)}}{\frac{\pi_1(Y)}{[\pi_1(Y), \pi_1(Y)]}}$$

Example 2: Let $G = \pi_1(X)$, $E = \pi_1(Y)$ then Stallings shows that for each $n \geq 0$,

$$f_*: G/G_n \xrightarrow{\ \cong\ } E/E_n$$

$\{G_n\}$ = lower central series of G:

$$G_1 = G \quad , \quad G_n = [G_{n-1}, G].$$

Theorem (Stallings): Let $\phi: G \longrightarrow E$ be a homomorphism of groups that induces $\cong$ on H_1 and an epimorphism on H_2. Then for any finite n, ϕ induces iso $\phi_*: \frac{G}{G_n} \xrightarrow{\cong} \frac{E}{E_n}$.

Recall by $H_*(G)$ we mean $H_*(K(G,1))$

Q. What about the derived series?

Recall, derived series:

$$G^{(0)} = G$$
$$G^{(n+1)} = [G^{(n)}, G^{(n)}]$$

A. $G/G^{(n)}$ is not necessarily preserved under homology equivalence.

Example: Let K be a knot in S^3, with $\Delta_K \neq 1$, $G = \pi_1(S^3 - K)$

$$\phi : G \twoheadrightarrow \mathbb{Z} \quad \text{abelianization}$$

(1) ϕ_* $\cong$ on homology

(2) [Cochran] $G/G^{(n)}$ is "large" $\left(G^{(n)}/G^{(n+1)} \neq 1\right)$

(3) $\mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z}$ $\forall\, n \geq 1$.

$$\therefore \ \phi_* : \frac{G}{G^{(n)}} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}^{(n)} = \mathbb{Z} \quad \text{not mono!}$$

Also,

meridian : $\mathbb{Z} \longrightarrow G$

$\Rightarrow$ $(\text{meridian})_* : \mathbb{Z} \hookrightarrow G/G^{(n)}$ not surjective!

<u>Theorem</u> (Cochran-H): If $\phi: F \to B$ induces a monomorphism on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$, F free group (<u>not</u> necessarily f.g.), B finitely relation then $\forall\, n \geq 1$

$$\phi_*: F/F^{(n)} \hookrightarrow B/B^{(n)}.$$

We can get a stronger result if we use a refined version of derived series!

Torsion-Free Derived Series : $G_H^{(n)}$

(1) $G_H^{(0)} := G$

(2) Assume (i) $G_H^{(n)}$ has been defined,
(ii) $G_H^{(n)} \triangleleft G$ (iii) $\mathbb{Z}[G/G_H^{(n)}]$ is an Ore domain

$\Rightarrow$ By (2) $\mathbb{Z}[G/G_H^{(n)}] \hookrightarrow \mathcal{K}_n$ = classical right ring of quotients

Consider:

$$G_H^{(n)} \xrightarrow{\alpha_n} \underbrace{\frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]}}_{} \xrightarrow{B_n} \frac{G_H^{(n)}}{[G_H^{(n)}, G_H^{(n)}]} \otimes_{\mathbb{Z}G/G_H^{(n)}} \mathcal{K}_n$$

right $\mathbb{Z}[G/G_H^{(n)}]$ module by
$\sigma g = g^{-1}\sigma g$ for $g \in G$.

(3) $G_H^{(n+1)} := \ker(B_n \circ \alpha_n) = \alpha_n^{-1}\left(\text{Torsion submodule of } G_H^{(n)}/[G_H^{(n)}, G_H^{(n)}]\right)$

$\Rightarrow$ Immediate that $G_H^{(n+1)} \triangleleft G_H^{(n)}$.
Can show $G_H^{(n+1)} \triangleleft G$ and
$\mathbb{Z}[G/G_H^{(n+1)}]$ ORE ring.

☆ $G_H^{(n)}/G_H^{(n+1)} = H_1(G_H^{(n)};\mathbb{Z})/\mathbb{Z}\tfrac{G}{G_H^{(n)}}$-torsion.

Examples:

(1) F free group

Since $F^{(n)}/F^{(n+1)}$ is torsion-free as $\mathbb{Z}[F/F^{(n)}]$-module,

$$\boxed{F_H^{(n)} = F^{(n)}} \quad \forall\ n \geq 0.$$

(2) K knot in S^3, $G = \pi_1(S^3 - K)$

Since $G^{(1)}/G^{(2)}$ = Alex. module is a $\mathbb{Z}[G/G^{(1)}]$-torsion module,

$G_H^{(n)} = [G, G]$ $\forall\ n \geq 1$. Hence

$$\boxed{G/G_H^{(n)} \cong \mathbb{Z}} \quad \forall\ n \geq 1.$$

* $\{G_H^{(n)}\}$ is a characteristic but not totally invariant series of G!

However, we have the following:

Proposition (Cochran-H): If $\phi: A \longrightarrow B$ induces a monomorphism on

$$\frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$ then

$$\phi(A_H^{(n+1)}) \subset B_H^{(n+1)}.$$

In particular, we have homomorphism

$$\phi_*: \frac{A}{A_H^{(n+1)}} \longrightarrow \frac{B}{B_H^{(n+1)}}.$$

Theorem (Cochran-H): If $\phi: A \to B$ is a mono. on $H_1(-;\mathbb{Q})$ and an epi. on $H_2(-;\mathbb{Q})$, A finitely generated, B finitely related then $\forall n \geq 1$,

$$\phi_* : \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}.$$

If ϕ onto then ϕ_* (as above) is $\cong$.

Corollary: If L is a boundary link then $G \twoheadrightarrow F$ induces

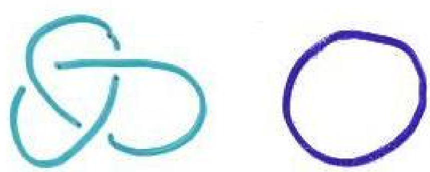
$$G/G_H^{(n)} \xrightarrow{\cong} F/F^{(n)}.$$

Recall: L is a boundary link if L bounds disjoint seifert surfaces

$\Rightarrow \pi_1(S^3 - L) \twoheadrightarrow$ Free group of rank m

$m = \#$ components of L

Ex.



$m=2$.

$G = \pi_1(S^3 - L) \twoheadrightarrow F(2)$

Even though $G/G_H^{(n)}$ is not invariant under homology cobordism, we can still use the previous theorem to get new homology cobordism invariants! To do this, we use Cheeger-Gromov von Neumann ρ-invt.

- (M^3, g) closed Riemannian mfld
  $\phi : \pi_1(M) \longrightarrow \Gamma$
  $$\rho^{(2)}_\Gamma(M) := \eta^{(2)}_\Gamma(M, g) - \eta_0(M, g)$$

Thm (Cheeger-Gromov): $\rho^{(2)}_\Gamma(M)$ is independent of g.

- M^3 closed, $G = \pi_1(M)$
  $\boxed{\Gamma_n := G/G_H^{(n+1)}}$ and $\phi_n : G \twoheadrightarrow \Gamma_n$.

Def: $\boxed{\rho^{(2)}_n(M) := \rho^{(2)}_{\Gamma_n}(M)}$

using Γ_n as above.

$\rho^{(2)}_\Gamma$-invts and signature defects

Let W^4 be a smooth 4-mfld (with or without boundary), $\phi : \pi_1(W) \longrightarrow \Gamma$ with Γ a PTFA (poly-torsion-free abelian) group.

<u>Note</u>: Γ PTFA means Γ solvable with torsion-free abelian quotients.

Γ PTFA $\Rightarrow \mathbb{Z}\Gamma \hookrightarrow \mathcal{K}$ right ring of quotients.

$$\begin{array}{ccc} \mathbb{Z}\Gamma & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{N}(\Gamma) & \longrightarrow & \mathcal{U}(\Gamma) \end{array}$$

$\mathcal{N}(\Gamma)$ = von Neumann Algebra

$\mathcal{U}(\Gamma)$ = unbounded operators affiliated with $\mathcal{N}(\Gamma)$

Von Neumann trace can be defined for $h \in \mathrm{Herm}_n(\mathcal{U}\Gamma)$. Let h be intersection form on $H_2(W;\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma$, $p_\pm$ char. function of $\mathbb{R}_\pm$

<u>Def</u>: $$\sigma^{(2)}_\Gamma(W) = \underbrace{tr_\Gamma\, p_+(h)}_{\text{dimension of + eigenspace}} - \underbrace{tr_\Gamma\, p_-(h)}_{\text{dimension of − eigenspace}}$$

$tr_\Gamma(h) = \langle h(e), e \rangle_{\ell^2\Gamma}$

Properties of $\rho^{(2)}$

(Γ- Induction)

P1: If $\Gamma \subset \tilde{\Gamma}$ then and $\phi : \pi_1(M) \to \Gamma$

then $\boxed{\rho^{(2)}_{\Gamma}(M) = \rho^{(2)}_{\tilde{\Gamma}}(M)}$

P2: If $M^3 = \partial W^4$ and $\phi : \pi_1(M) \longrightarrow \Gamma$

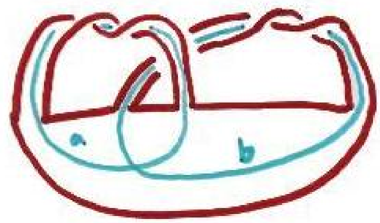
extends over W then

$$\boxed{\rho^{(2)}_{\Gamma}(M) = \sigma^{(2)}_{\Gamma}(W) - \sigma(W)}$$

This follows from L^2-index theorems for mflds with and without boundary.

Important Example

$(n=0)$ K knot in S^3, $M_K = 0$-surgery on K



$$V = \begin{pmatrix} lk(a,a^+) & lk(a,b^+) \\ lk(b,a^+) & lk(b,b^+) \end{pmatrix}$$

= (integral) Seifert matrix

for every $\omega \in S^1$, $\omega V + \bar{\omega} V^T$ is hermitian
$\Rightarrow$ get signatures.

$$\rho_0^{(2)}(M_K) = \rho^{(2)}(M_K ; \pi_1(M_K) \longrightarrow H_1(M_K) = \mathbb{Z})$$

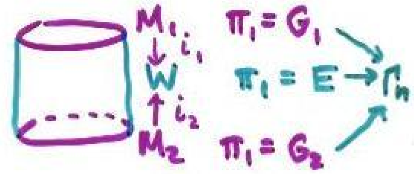
$$= \int_{\omega \in S^1} \sigma(\omega V + \bar{\omega} V^T)\, d\omega$$

$$\in \mathbb{R}$$

<u>Theorem(#)</u> $\rho_n^{(2)}(M)$ is an invariant of homology cobordism.

<u>Proof:</u> Let $M_1 \sim_H M_2$

$(i_j)_* \cong$ on H_*



① $\Rightarrow$ By thm, $G_i / (G_i)_H^{(n+1)} \hookrightarrow E / E_H^{(n+1)} =: \Gamma_n$

② By Γ-induction, $\rho_n^{(2)}(M_i) = \rho^{(2)}(M_i, G_i \to \Gamma_n)$

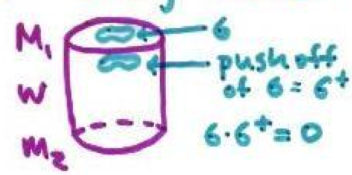
so since we have $\pi_1(W) = E \to \Gamma_n$

③ $\rho^{(2)}(M_1, G_1 \to \Gamma_n) - \rho^{(2)}(M_2, G_2 \to \Gamma_n) = \sigma^{(2)}_{\Gamma_n}(W) - \sigma(W)$

④ By hyp $H_2(M_1) \xrightarrow{\cong} H_2(W)$ so every $\sigma \in H_2(W)$ comes from $H_2(\partial W)$.

But if $\alpha, \beta \in H_2(\partial W)$, $\alpha \cdot \beta = 0$

$\Rightarrow \sigma(W) = 0$



⑤ By homological args, we can show

$H_2(M_1; \mathcal{U}\Gamma_n) \twoheadrightarrow H_2(W; \mathcal{U}\Gamma_n)$,

use arg as above to show $\sigma^{(2)}_{\Gamma_n}(W) = 0$.

$\therefore \rho_n^{(2)}(M_1) - \rho_n^{(2)}(M_2) = 0$!

Theorem (H): For each $n \geq 0$, the image of $\rho_n^{(2)}$ is dense in $\mathbb{R}$ and is infinitely generated.

Construction of examples:

Let $M = \#_{i=1}^{k} S^1 \times S^2$, $F = \pi_1(M) =$ free group, rank k.

Recall $F_H^{(n)}/F_H^{(n+1)} = F^{(n)}/F^{(n+1)} \neq 1$.

Let K be a knot in S^3

$1 \neq \eta \in F^{(n)} - F^{(n+1)}$

$$M(\eta, K) = (M - (\eta \times D^2)) \cup_\varphi (S^3 - K)$$

$\varphi : \partial(S^3 - K) \to \partial(\eta \times D^2)$ $\quad \varphi : \mu_K \longmapsto \ell_\eta^{-1}$

$$\ell_K \longmapsto \mu_\eta$$

$$\rho_k^{(2)}(M(\eta, K)) = \begin{cases} 0 & k < n \\ \rho_0^{(2)}(K) & k \geq n \end{cases}$$

Recall $\rho_0^{(2)}(K) = \int_{S^1} \sigma(\omega V + \bar{\omega} V^T)\, d\omega$

Can show $\rho_0^{(2)}$ dense in $\mathbb{R}$ and is infinitely generated subgroup of $\mathbb{R}$. (J.C. Cha - Livingston, Cochran - Orr - Teichner)

$\boxed{n=1}$

(*)

$\eta=[x,y] \in F^{(1)}-F^{(2)}$

$\xrightarrow{\text{tie in } K}$

$M = S^1\times S^2 \# S^1\times S^2$

$\pi_1(M)=F_2=\langle x,y\rangle$

$M' = M(\eta,K)$

$\rho_0(M')=0 \quad \rho_\ell(M')=\rho_0(K) \;\; \ell\geq 1$

K= left handed trefoil

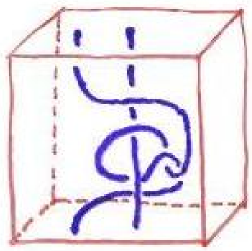
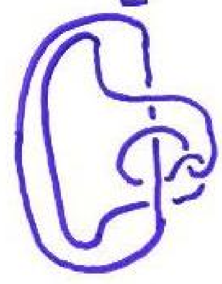
M'

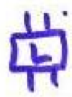
$\rho_0(M')=0$

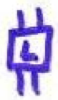
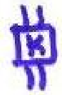
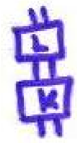
$\rho_\ell(M')=4/3 \quad \ell\geq 1$

Let $C(m) = \left\{ \begin{array}{l} \text{concordance group of} \\ m\text{-component } \underline{\text{string}}\ \underline{\text{links}} \end{array} \right\}$.

<u>String link</u> L $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \hat{L}$

 $\xrightarrow{\text{closure of } L}$ 

"<u>Add</u>":  +  =  ~ "connected sum of links"

↑ (nonabelian)

We have filtration of $C(m)$

$$\cdots \mathcal{F}_{(n.5)}(m) \subset \mathcal{F}_{(n)}(m) \subset \cdots \subset \mathcal{F}_{(0.5)}(m) \subset \mathcal{F}_{(0)}(m) \subset C(m)$$

by $\mathcal{F}_{(n)}(m) = (n)$-solvable string links.

<u>Proposition</u> (H): If $L \in \mathcal{F}_{(n.5)}$ then $\rho_n(L) = 0$.

Theorem (H): For each $m \geq 2$ and $n \geq 1$,

$$\mathcal{F}_{(n)}(m) / \mathcal{F}_{(n.5)}(m)$$

is infinitely generated.

Proof: Let L be trivial link w/ m components.
$\Rightarrow \pi_1(S^3 - L) = F$. If $m \geq 2, n \geq 1$, $F^{(n)}/F^{(n+1)} \neq 0$.
Choose $\eta \in F^{(n)} - F^{(n+1)}$ as before. Obtain link $\{L_i\}$ by tying strands into knot K.

- Each L_i is (n)-solvable ((Cochran-Orr-Teichner)
- L_i is not $(n.5)$-solvable if $\rho_n(L_i) \neq 0$.
- $\rho_n(\{L_i\})$ is infinitely generated from before
- Moreover, each L_i is a boundary link

Prop(H): ρ_n is additive on class of boundary links!

$\blacksquare$