

RICE UNIVERSITY

**The  $(n)$ -Solvable Filtration of the Link  
Concordance Group and Milnor's  $\bar{\mu}$ -Invariants**

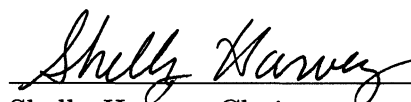
by

**Carolyn Ann Otto**

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APPROVED, THESIS COMMITTEE:



---

Shelly Harvey, Chair  
Associate Professor of Mathematics



---

Tim D. Cochran  
Professor of Mathematics



---

Liliana Borcea  
Noah G. Harding Professor of  
Computational and Applied Mathematics

Houston, Texas

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# Abstract

The  $(n)$ -solvable filtration of the link concordance group and

Milnor's  $\bar{\mu}$ -invariants

by

Carolyn Otto

We establish several new results about the  $(n)$ -solvable filtration,  $\{\mathcal{F}_n^m\}$ , of the string link concordance group  $\mathcal{C}^m$ . We first establish a relationship between  $(n)$ -solvability of a link and its Milnor's  $\bar{\mu}$ -invariants. We study the effects of the Bing doubling operator on  $(n)$ -solvability. Using this results, we show that the “other half” of the filtration, namely  $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ , is nontrivial and contains an infinite cyclic subgroup for links with sufficiently many components. We will also show that links modulo  $(1)$ -solvability is a nonabelian group. Lastly, we prove that the Grope filtration,  $\mathcal{G}_n^m$  of  $\mathcal{C}^m$  is not the same as the  $(n)$ -solvable filtration.

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# Chapter 1

## Introduction

### 1.1 Background

A **link**  $L$  (with  $m$  components) is an embedding  $L : \coprod_m S^1 \hookrightarrow S^3$ . A link with one component is called a **knot**. In the late 1950's, Fox and Milnor introduced the idea that the concordance classes of links and that these classes were an obstruction to the removal of a link singularity. In doing so, they investigated the notion of a knot being slice. In other words, whether or not a knot bounds a smooth disk in  $B^4$ . If the link of a singularity is slice, then we can replace this singularity with a smooth disk.

An equivalence relation on knots in  $S^3$  can be defined by using slice knots:  $K \sim J$  if  $K \# -J$  is slice. The knot concordance group  $\mathcal{C}$  was introduced by Fox and Milnor in 1966 [FM66] using this equivalence on the set of knots. Two knots,  $K$  and  $J$  are said to be *concordant* if  $K \times \{0\}$  and  $J \times \{1\}$  cobound a smoothly embedded annulus in  $S^3 \times [0, 1]$ . If a knot is slice, it is in the identity class of this group. This abelian



group is a well studied object in low-dimensional topology, however there is much that is still unknown about the structure of  $\mathcal{C}$ . Thus  $\mathcal{C}$  has remained an active object of study since its introduction.

Here we are particularly interested in the group of concordance classes of string links called the string link concordances group, which is denoted  $\mathcal{C}^m$ .

In order to investigate the structure of this group, Cochran, Orr and Teichner introduced two filtrations of this group: the  $(n)$ -solvable filtration,  $\{\mathcal{F}_n^m\}$  and the Grope filtration,  $\{\mathcal{G}_n^m\}$  [COT03]. The notion of  $(n)$ -solvability can be thought of as an algebraic approximation to a link being slice (or “0” in  $\mathcal{C}^m$ ). Gropes, on the other hand, are more geometric in nature and can be thought of as geometric approximations to slicing disks. In [COT03], it was shown that these two filtrations are related for all  $n \in \mathbb{N}$  and  $m \geq 1$  by  $\mathcal{G}_{n+2}^m \subseteq \mathcal{F}_n^m$ .

Much work has been done in the quest of understanding the  $(n)$ -solvable filtration. In particular, many have studied successive quotients of this filtration and some of their contributions can be found in [Cha10], [CH08], [CHL09], [Har08].

For example, Harvey first showed that  $\mathcal{F}_n^m / \mathcal{F}_{n+1}^m$  is a nontrivial group that contains an infinitely generated subgroup [Har08]. She also showed that this subgroup is generated by boundary links (links with components that bound disjoint Seifert surfaces). Cochran and Harvey generalized this result by showing that  $\mathcal{F}_n^m / \mathcal{F}_{n.5}^m$  contains an infinitely generated subgroup [CH08]. Again, this subgroup consists entirely of boundary links.

Since  $\mathcal{C}^m$  is a nonabelian group for  $m \geq 2$  [LD88], questions were raised about whether or not successive quotients of  $\{\mathcal{F}_n^m\}$  would be abelian. Let  $\mathcal{F}_{-0.5}^m$  denote the

set of (string) links that have all pairwise linking numbers equal to zero. It is known that  $\mathcal{F}_{-0.5}^m/\mathcal{F}_0^m$  is an abelian group while  $\mathcal{C}^m/\mathcal{F}_0^m$  is nonabelian. For quotients of the higher terms in the filtration, it is unknown whether or not they are abelian.

## 1.2 Summary of Results

Let  $L$  be an  $m$ -component link in  $S^3$  and  $G = \pi_1(S^3 - L)$ . The  $n^{\text{th}}$  term of the lower central series of a group  $G$ , denoted  $G_n$ , is inductively defined by  $G_1 = G$  and  $G_n = [G_{n-1}, G]$ , where the latter group is generated by elements of the form  $aba^{-1}b^{-1}$  for  $a \in G_{n-1}$  and  $b \in G$ . Milnor invariants, denoted  $\bar{\mu}$ , can be thought of as “higher-order” linking numbers between components of a link ([Mil54], [Mil57]). These are known to be invariant under concordance and measure how deeply the longitudes of each component of a link  $L$  lie in the lower central series of the link group  $G$ .

Up to this point, little has been known about the relationship of Milnor’s invariants and  $(n)$ -solvability. We establish the following relationship.

**Theorem 3.7.** *If  $L$  is an  $(n)$ -solvable link with  $m$  components, then  $\bar{\mu}_L(I) = 0$  for  $|I| \leq 2^{n+2} - 1$ .*

In other words, if a link is  $(n)$ -solvable, then all of its  $\bar{\mu}$ -invariants will vanish for lengths less than or equal to  $2^{n+2} - 1$ . Moreover, this theorem is sharp in the sense that we exhibit  $(n)$ -solvable links with  $\bar{\mu}(I) \neq 0$  for  $|I| = 2^{n+2}$ .

A common “doubling operation” of links is Bing doubling, depicted in Figure 1.1. This operator doubles the number of components of the original link.

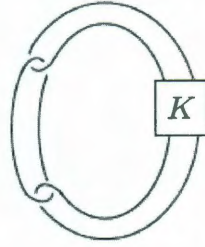


Figure 1.1: The Bing double of a knot  $K$ ,  $BD(K)$

To Bing double a link, we perform this operation on each component. Bing doubling a knot always gives a boundary link so the Milnor's invariants vanish for all lengths. On the contrary, Bing doubling a link with nonvanishing  $\bar{\mu}$  will never give a boundary link. We study the effects of Bing doubling on  $(n)$ -solvability. We show that solvability is not only preserved under this operator, but it increases the solvability by one.

**Proposition 4.11.** *If  $L$  is an  $(n)$ -solvable link, then  $BD(L)$  is  $(n + 1)$ -solvable. Moreover, if  $L$  is an  $(n.5)$ -solvable link, then  $BD(L)$  is  $((n + 1).5)$ -solvable.*

Until this point, nothing was known about the “other half” of the filtration,  $\mathcal{F}_{n.5}^m / \mathcal{F}_{n+1}^m$ . Using the above results, we show that the “other half” of the  $(n)$ -solvable filtration is nontrivial.

**Theorem 5.1.**  $\mathcal{F}_{n.5}^m / \mathcal{F}_{n+1}^m$  contains an infinite cyclic subgroup for  $m \geq 3 * 2^{n+1}$ .

The examples used come from iterated Bing doubles of links with nonvanishing  $\bar{\mu}$ -invariants. Thus, our examples are not concordant to boundary links, so the subgroups that they will generate will be different than those previously detected. The result of Theorem 5.1 is still unknown for knots.

Since the knot concordance group,  $\mathcal{C}$ , is abelian, all successive quotients of the  $(n)$ -solvable filtration are abelian. However, it is known that  $\mathcal{C}^m$  is nonabelian for  $m \geq 2$  [LD88]. We have shown that certain successive quotients are not abelian.

**Theorem 5.3.**  $\mathcal{F}_{-0.5}^m/\mathcal{F}_1^m$  is nonabelian for  $m \geq 3$ .

Similar to the relationship between  $(n)$ -solvability and  $\bar{\mu}$ -invariants, we establish a relationship between  $\bar{\mu}$ -invariants and a link in which all of its components bound disjoint gropes of height  $n$ . This relationship says that if all components of a link bound disjoint gropes of a certain height, then its  $\bar{\mu}$  invariants vanish for certain lengths.

**Corollary 6.7.** A link  $L$  with components that bound disjoint Gropes of height  $n$  has  $\bar{\mu}_L(I) = 0$  for  $|I| \leq 2^n$ .

A result of Lin [Lin91] states that  $k$ -cobordant links will have the same  $\bar{\mu}$ -invariants. Using this result, the proof of this proposition relies on showing that  $L$  is  $2^{n+1}$ -cobordant to a slice link.

The two filtrations are related by the fact that  $\mathcal{G}_{n+2}^m \subseteq \mathcal{F}_n^m$  for all  $n \in \mathbb{N}$  and  $m \geq 1$  [COT03]. A natural question is whether or not these filtrations are actually the same. We show that these filtrations differ at each stage.

**Corollary 6.9.**  $\mathcal{F}_n^m/\mathcal{G}_{n+2}^m$  is nontrivial for  $m \geq 2^{n+2}$ . Moreover,  $\mathbb{Z} \subset \mathcal{F}_n^m/\mathcal{G}_{n+2}^m$ .

### 1.3 Outline of Thesis

In Chapter 2 we review the string link concordance group,  $\mathcal{C}^m$  and the  $(n)$ -solvable filtration,  $\{\mathcal{F}_n^m\}$  defined by Cochran, Orr and Teichner. This filtration is very algebraic in nature. We also demonstrate properties of links in this filtration, known as  $(n)$ -solvable links.

In Chapter 3 we give the definition of Milnor's  $\bar{\mu}$ -invariants of links. We discuss when these invariants vanish and when they are additive. We establish a relationship between  $\bar{\mu}$ -invariants and  $(n)$ -solvability.

In Chapter 4 we examine the definition of Bing doubling using infection by a string link. We investigate the effects of Bing doubling on links in various levels of the  $(n)$ -solvable filtration.

In Chapter 5 we give applications of the relationship between  $\bar{\mu}$  and  $\{\mathcal{F}_n^m\}$  found in Chapter 3. This chapter focuses on the structure of successive quotients of  $\{\mathcal{F}_n^m\}$ . We show nontriviality in one such quotient and investigate the commutivity in other quotients using results from previous chapters.

In Chapter 6 we define another filtration of  $\mathcal{C}^m$  known as the Grope filtration,  $\{\mathcal{G}_n^m\}$  which is more geometric than  $\{\mathcal{F}_n^m\}$ . We establish a relationship between  $\bar{\mu}$ -invariants and links in this filtration. Finally, we showed that these two filtrations are different by showing that the quotient  $\mathcal{F}_n^m/\mathcal{G}_n^m$  is nontrivial for certain  $m$ .

# Chapter 2

## $(n)$ -Solvable Filtration

### 2.1 String Link Concordance Group

A **knot** is an embedding  $S^1 \hookrightarrow S^3$ . Two knots,  $K$  and  $J$  are said to be *concordant* if  $K \times \{0\}$  and  $J \times \{1\}$  cobound a smoothly embedded annulus in  $S^3 \times [0, 1]$ . The set of knots modulo concordance forms a group under the operation of connected sum, known as the knot concordance group  $\mathcal{C}$ . This group is known to be an abelian group. A link is a generalization of a knot in which it may have more components. More specifically, an  $m$ -**component link** is an embedding  $\coprod_m S^1 \hookrightarrow S^3$ . It is apparent that a link of one component is precisely a knot. The connected sum operation is not well defined for links. Therefore, in order to define a group structure on links, it is necessary to study string links. We will give the definition of string links stated by Habegger and Lin in [HL90].

**Definition 2.1.** Let  $D$  be the unit disk,  $I$  the unit interval and  $\{p_1, p_2, \dots, p_k\}$  be  $k$  points in the interior of  $D$ . A  $k$ -**component string link** is a smooth proper

embedding  $\sigma : \coprod_{i=1}^k I_i \rightarrow D \times I$  such that

$$\sigma|_{I_i}(0) = \{p_i\} \times \{0\};$$

$$\sigma|_{I_i}(1) = \{p_i\} \times \{1\}.$$

The image of  $I_i$  is called the  $i^{\text{th}}$  *string* of the string link. An orientation on  $\sigma$  is induced by the orientation of  $I$ . Two string links  $\sigma$  and  $\sigma'$  are said to be *equivalent* or *isotopic* if there is an isotopy  $h : D^2 \times I \rightarrow D^2 \times I$  such that  $h$  fixes the boundary and  $h(\sigma) = \sigma'$ .

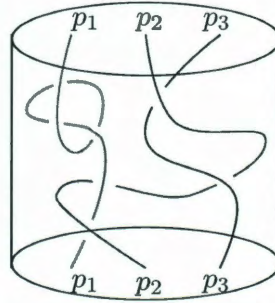


Figure 2.1: An example of a three component string link

The operation on string links is the stacking operation seen in the braid group. If  $L_1$  and  $L_2$  are in  $\mathcal{C}^m$ , then  $L_1L_2$  is the string link obtained by stacking  $L_1$  on top of  $L_2$ .

The notion of concordance can be generalized for string links, see Figure 2.2.

**Definition 2.2.** Two string links,  $\sigma_1, \sigma_2$  ( $m$ -component) are **concordant** if there exists a smooth embedding  $H : \coprod_m (I \times I) \rightarrow B^3 \times I$  that is transverse to the boundary and such that  $H|_{\coprod_m I \times \{0\}} = \sigma_1$ ,  $H|_{\coprod_m I \times \{1\}} = \sigma_2$ , and  $H|_{\coprod_m \partial I \times I} = j_0 \times id_I$  where  $j_0 : \coprod_m \partial I \rightarrow S^2$ .

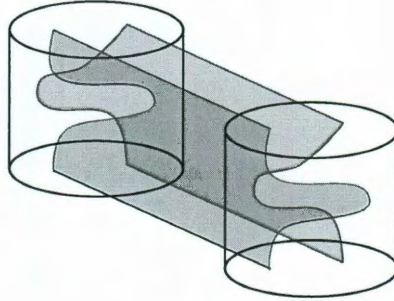


Figure 2.2: String link concordance

Under the operation of stacking, the concordance classes of  $m$ -component string links form a group, denoted  $\mathcal{C}^m$ , and is known as the string link concordance group. The identity class of this group is the class of slice string links (string links which are string link concordant to the trivial string link). The inverses are the string links obtained by reflecting the string link about  $D \times \{1/2\}$  and reversing the orientation. When  $m = 1$ ,  $\mathcal{C}^m$  is the knot concordance group. For  $m \geq 2$ , it has been shown that  $\mathcal{C}^m$  is not abelian [LD88].

If  $L$  is a string link, the closure of  $L$ , denoted  $\hat{L}$ , is the ordered, oriented link in  $S^3$  obtained by gluing  $\partial D^2 \times I$  to  $\partial D^2 \times I$  of the standard trivial string link. This gives a canonical way to obtain a link from a string link. If two string links are concordant, then their closures are concordant as links. This means that the closure of a slice string link is a slice link, or that it bounds a smooth disk in  $D^4$ .

Every link has a string link representative, as seen in the following lemma by Habegger and Lin [HL90].

**Lemma 2.3** (Habegger-Lin). *Given any link,  $L$  in  $S^3$ , there exists a string link  $\sigma$  such that  $\hat{\sigma}$  is isotopic to  $L$ .*

The terminology given by Habegger and Lin in their work [HL90] will be followed



in order to prove the lemma.

**Definition 2.4.** A  $d$ -base for a link is a disk embedded in  $S^3$  such that it intersects each link component transversely exactly once with consistent intersection numbers (see Figure 2.3(b) for example).

*Proof.* Given a link  $L$ , find any  $d$ -base for the link (the choice of  $d$ -base is not unique). By taking a neighborhood of this disk, we obtain a  $D^2 \times I$  that contains a trivial string link. The exterior of the trivial string link, say  $\sigma$ , will then be a string link such that  $\hat{\sigma}$  is isotopic to  $L$ .  $\square$

The following example in Figure 2.3 illustrates the procedure in the proof of the lemma.

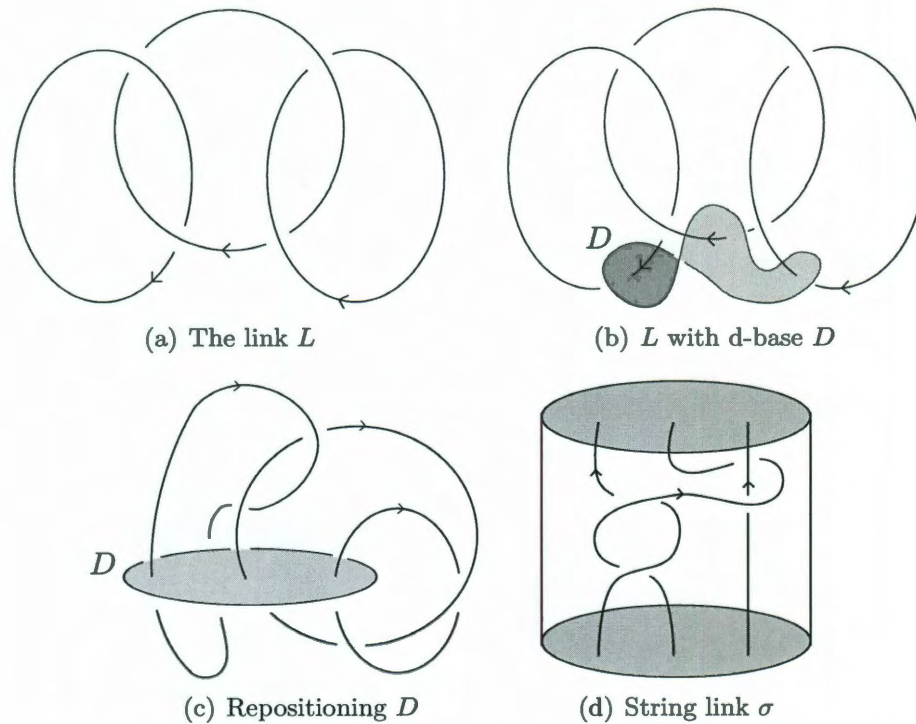


Figure 2.3: An illustration of the procedure of obtaining a string link representative of a link

## 2.2 The $(n)$ -Solvable Filtration

A **filtration** of a group is a nested sequence of normal subgroups. Often, filtrations of a group are used to study the structure of that group. In order to study the structure of  $\mathcal{C}^m$ , Cochran, Orr and Teichner [COT03] defined the  $(n)$ -solvable filtration,  $\{\mathcal{F}_n^m\}$ ,

$$\{0\} \subset \cdots \subset \mathcal{F}_{n+1}^m \subset \mathcal{F}_{n.5}^m \subset \mathcal{F}_n^m \subset \cdots \subset \mathcal{F}_{0.5}^m \subset \mathcal{F}_0^m \subset \mathcal{C}^m.$$

Before the definition of  $(n)$ -solvability is given, we recall a few definitions.

**Definition 2.5.** Let  $G^{(i)}$  denote the  $i^{\text{th}}$  term of the derived series of a group  $G$  that is inductively defined by  $G^{(0)} := G$  and  $G^{(i+1)} := [G^{(i)}, G^{(i)}]$ . The latter group is generated by elements of the form  $aba^{-1}b^{-1}$  for  $a, b \in G^{(i)}$ .

**Definition 2.6.** Let  $L$  be a link an  $m$ -component link in  $S^3$ . The **zero-framed surgery of  $L$** , denoted  $M_L$  is given by

$$M_L = (S^3 - N(L)) \cup_h [(S^1 \times D^2) \cup \cdots \cup (S^1 \times D^2)]$$

where  $h : \cup \partial(S^1 \times D^2) \rightarrow \partial(S^3 - N(L))$  is the map which sends the meridian  $\mu_i = \{pt\} \times D^2$  to the  $i^{\text{th}}$  longitude  $l_i$  of  $S^3 - N(L)$ .

**Definition 2.7.** An  $m$ -component link  $L$  is  $(n)$ -solvable if the zero-framed surgery,  $M_L$ , bounds a compact, smooth 4-manifold,  $W^4$ , such that the following hold:

i)  $H_1(M_L; \mathbb{Z}) \cong \mathbb{Z}^m$  and  $H_1(M_L) \rightarrow H_1(W; \mathbb{Z})$  is an isomorphism induced by the inclusion map;

ii)  $H_2(W; \mathbb{Z})$  has a basis consisting of connected, compact, oriented surfaces,  $\{L_i, D_i\}_{i=1}^r$ , embedded in  $W$  with trivial normal bundles, wherein the surfaces are pairwise disjoint except that, for each  $i$ ,  $L_i$  intersects  $D_i$  transversely once with positive sign;

iii) For all  $i$ ,  $\pi_1(L_i) \subset \pi_1(W)^{(n)}$  and  $\pi_1(D_i) \subset \pi_1(W)^{(n)}$ .

The manifold  $W$  is called an  $(n)$ -**solution** for  $L$  and a string link is  $(n)$ -**solvable** if its closure in  $S^3$  is an  $(n)$ -solvable link.

A link is  $(n.5)$ -**solvable** if, in addition to the above,  $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$  for all  $i$ . In this instance,  $W$  is called an  $(n.5)$ -**solution** for  $L$ .

The notion of  $(n)$ -solvability can be thought of as an algebraic approximation to a link being slice, or the identity in  $\mathcal{C}^m$ .

For all  $m \geq 1$ , the  $(n)$ -solvable filtration of the string link concordance group is defined by setting  $\mathcal{F}_n^m$  to be the set of  $(n)$ -solvable links,  $L$  in  $S^3$  for  $n \in \frac{1}{2}\mathbb{N}_0$ . It is known that  $\mathcal{F}_n^m$  is a normal subgroup of  $\mathcal{C}^m$  for all  $m \geq 1$  and  $n \in \frac{1}{2}\mathbb{N}_0$ . For convenience's sake,  $\mathcal{F}_{-0.5}^m$  will denote the set of links with all pairwise linking numbers equal to zero. It is worth noting that if  $L \in \mathcal{C}^m$ , then  $L$  and  $\hat{L}$  have all the same pairwise linking numbers.

## 2.3 Properties of $(n)$ -Solvable Links

In this section, we will give two properties of  $(n)$ -solvable links that will be used in later chapters.

**Proposition 2.8.** *If  $L$  is an  $(n)$ -solvable link, then every sublink of  $L$  is an  $(n)$ -*

*solvable link.*

*Proof.* Suppose  $L = L_1 \cup L_2 \cup \cdots \cup L_m$  is an  $(n)$ -solvable,  $m$ -component link and let  $W$  be its  $(n)$ -solution. Consider the sublink of  $L$ ,  $J = L_2 \cup \cdots \cup L_m$ , obtained by omitting the  $L_1$  component. We will create a cobordism,  $X$ , between  $M_J$  and  $M_L$  such that  $X \cup W$  is an  $(n)$ -solution for  $J$ .

Consider a new link,  $L'$ , obtained from  $L$  by performing a zero-framed surgery on the meridian,  $\mu$  of  $L_1$ , which we will refer to as a “helper circle” (see Figure 2.4(a)). By performing handle slides, we see that the new 3-manifold is the same as the one in Figure 2.4(b). Since zero-framed surgery on the Hopf link is homeomorphic to  $S^3$ , zero-framed surgery on  $L'$  is in fact the manifold  $M_J$ .

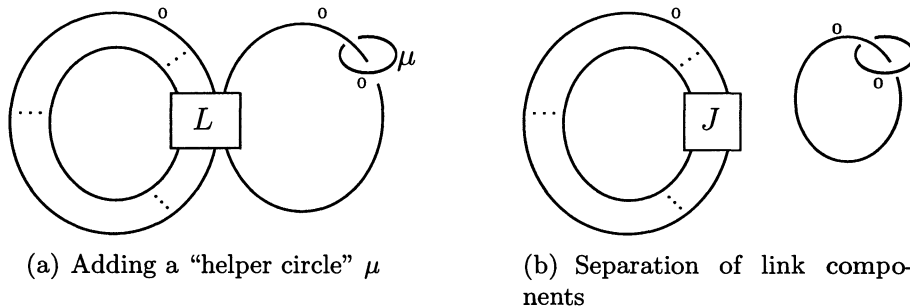


Figure 2.4: A “helper circle” added to  $L$  and the separation of the components resulting in a new link  $J$

By adding the “helper circle” to  $L'$ , we are adding a 2-handle,  $D = D^2 \times D^2$ , onto the boundary of  $M_L \times [0, 1]$ . The 4-manifold obtained by attaching  $D$  to  $M_L \times [0, 1]$  is a cobordism between  $M_J$  and  $M_L$ . We notice that  $W \cup (M_L \times [0, 1]) \cup D \cong W \cup D$ . We aim to show that  $W \cup D$  is an  $(n)$ -solution for  $J$ . To see this, consider the effect on homology and  $\pi_1$  of adding  $D$  to  $M_L \times [0, 1]$ .

Following the definition, we must first check that  $H_1(M_J)$  is isomorphic to  $H_1(W \cup$

$D) \cong \mathbb{Z}^{m-1}$  induced by inclusion. Since  $W$  is an  $(n)$ -solution for  $L$ ,  $H_1(M_L)$  is isomorphic to  $H_1(W) \cong \mathbb{Z}^m$  which is induced by inclusion. Adding a 2-handle to  $W$  kills off a generator of  $H_1(W)$  since we are attaching along an element of infinite order. Thus  $H_1(W \cup D) \cong \mathbb{Z}^{m-1}$ .

The manifold  $W \cup D$  is obtained (up to homotopy equivalence) by adding a 2-cell to  $W$  along  $\mu$ . The meridian  $\mu$  has infinite order in  $H_1(W)$  and so  $H_2(W \cup D) \cong H_2(W)$ . Since  $W$  is an  $(n)$ -solution for  $L$ , an appropriate basis for  $H_2(W \cup D)$  is obtained.

Lastly, the inclusion map  $i : W \hookrightarrow W \cup D$  gives  $i_*(\pi_1(W)^{(n)}) \subseteq \pi_1(W \cup D)^{(n)}$ . Since no elements were added to the basis of  $H_2(W \cup D)$ , it has the same basis as  $H_2(W)$ . Thus  $\pi_1(L_i) \subset \pi_1(W)^{(n)} \subseteq \pi_1(X \cup W)^{(n)}$  and similarly for  $\pi_1(D_i)$ , where  $\{L_i, D_i\}$  is a basis for  $H_2(W)$ . Thus  $W \cup D$  is an  $(n)$ -solution for  $J$  and  $J$  is  $(n)$ -solvable.

To obtain that any sublink is  $(n)$ -solvable, we continue this procedure to remove all components not in the desired sublink.  $\square$

**Remark 2.9.** *The converse of this proposition is not true. For example, each component of the Hopf link is trivial and thus  $(0)$ -solvable, but  $M_{\text{Hopf link}} \cong S^3$ . Since  $H_1(S^3) = 0$ , this link cannot be  $(0)$ -solvable.*

Again, assume that  $L$  is a link in  $S^3$  with  $m$  components,  $L_1, \dots, L_m$ . Denote by  $b_{ij}(L)$  the band connect sum (or band sum) of  $L_i$  and  $L_j$ , where  $i \neq j$ . In other words,  $b_{ij}(L)$  denotes the resulting link from connect summing the  $i^{\text{th}}$  and  $j^{\text{th}}$  components of  $L$ .

**Proposition 2.10.** *If  $L$  is an  $(n)$ -solvable link, then  $b_{ij}(L)$  is  $(n)$ -solvable.*

*Proof.* We will construct an  $(n)$ -solution for  $b_{ij}(L)$ . To do this, we form a cobordism

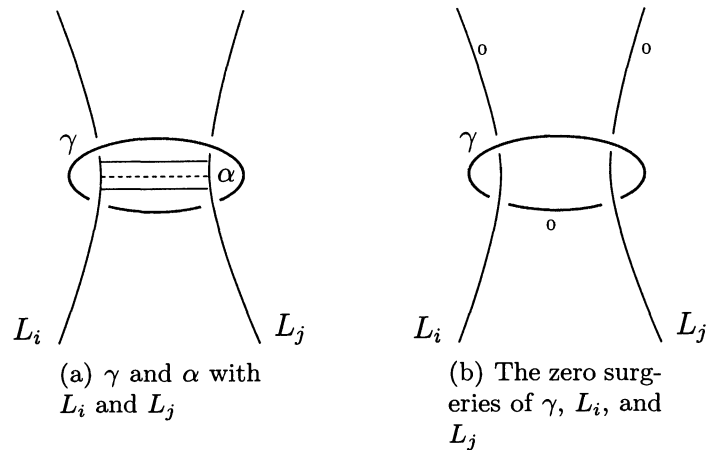


Figure 2.5: Band summing two link components

between  $M_L$  and  $M_{b_{ij}(L)}$ . First, consider the zero-framed surgery on  $L$ , denoted  $M_L$ . Let  $\alpha$  be a band connecting  $L_i$  and  $L_j$ . This band indicates where the band sum will be performed (see Figure 2.5(a)).

Let  $W$  be the 4-manifold obtained by taking  $M_L \times [0, 1]$  and adding a zero-framed 2-handle along  $\gamma$ , the curve indicated in Figure 2.5(a). Before we proceed, we need the following lemma.

**Lemma 2.11.**  $\partial W = M_L \amalg M_{b_{ij}(L)}$

*Proof.* It is apparent that  $M_L$  is part of the boundary. To see the other element of the boundary, consider Figure 2.5(b) as a 4-manifold diagram. We perform a handle slide by taking  $L_i$  and sliding it along  $L_j$  using the arc  $\alpha$ . As a result of this handle slide, we created a “helper circle” around  $L_j$  and thus can unknot and separate  $L_j$  from  $L$  (see Figure 2.6). This creates, as a 3-manifold, the zero-framed surgery on the Hopf link, which is homeomorphic to  $S^3$ . As a 3-manifold, we are left with  $M_{b_{ij}(L)}$ . Thus  $\partial W = M_L \amalg M_{b_{ij}(L)}$ .  $\square$

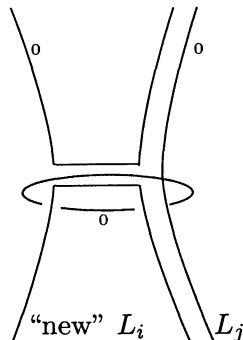


Figure 2.6: Performing a handle slide

We now continue with the proof of the proposition. Let  $X$  be an  $(n)$ -solution for  $L$ . Then  $H_1(M_L)$  is isomorphic to  $H_1(X) \cong \mathbb{Z}^m$ , induced by the inclusion map. Consider the manifold  $X \cup W$ . This manifold is obtained (up to homotopy equivalence) by adding a 2-cell to  $X$  along  $\gamma$ . Since  $\gamma$  has infinite order in  $H_1(\partial X) \cong H_1(X)$ ,  $H_1(X \cup W) \cong \mathbb{Z}^{m-1}$  and  $i : M_{b_{ij}(L)} \rightarrow X \cup W$  induces an isomorphism on  $H_1$ . Moreover,  $H_2(X) \cong H_2(X \cup W)$ . By assumption,  $H_2(X)$  has a basis  $\{L_i, D_i\}_{i=1}^r$  with  $L_i \cap D_j = \delta_{ij}$ . Since  $H_2$  is unchanged under this handle addition,  $H_2(X \cup W)$  has the same basis.

Finally, note that the condition on  $\pi_1$  is met by a completely analogous argument to the one in Proposition 2.8. Thus  $X \cup W$  is an  $(n)$ -solution for  $b_{ij}(L)$  and band summing preserves  $(n)$ -solvability.  $\square$

**Remark 2.12.** *The arguments in Propositions 2.8 and 2.10 can easily be generalized to give analogous results for  $(n,5)$ -solvable links.*

# Chapter 3

## Milnor's $\bar{\mu}$ -Invariants

### 3.1 Definition

In this section, we recall the definition of Milnor's  $\bar{\mu}$ -invariants. In the early 1950's, John Milnor defined a family of higher order linking numbers known as  $\bar{\mu}$ -invariants for oriented, ordered links in  $S^3$  [Mil54], [Mil57]. These numbers are not link invariants in the typical sense since there is some indeterminacy due to the choice of meridians of a link; however, as invariants of string links they are well defined [HL90]. In general, Milnor's invariants determine how deep the longitudes of each component lie in the lower central series of the link group. While Milnor's  $\bar{\mu}$ -invariants can be defined in several ways, we will focus on the definition centered around the Magnus expansion.

Suppose  $L$  is an  $m$ -component link in  $S^3$ . Let  $G = \pi_1(S^3 - L)$  be the fundamental group of the complement of  $L$  in  $S^3$ . The *lower central series* of  $G$ , denoted  $G_i$  is recursively defined by  $G_1 := G$  and  $G_i := [G_{i-1}, G]$ , where the latter group is generated by elements of the form  $aba^{-1}b^{-1}$  for  $a \in G_{i-1}$  and  $b \in G$ .



**Remark 3.1.** *The derived series and the lower central series of a group  $G$  are related by  $G^{(n)} \subset G_{2^n}$ . Since  $[G_r, G_s] \subseteq G_{r+s}$ , it is a straight forward computation to achieve the relation.*

Consider the nilpotent quotient group  $G/G_k$ . A presentation of this group, given by Milnor [Mil57], can be written

$$G/G_k \cong \langle \alpha_1, \alpha_2, \dots, \alpha_m | [\alpha_1, l_1], [\alpha_2, l_2], \dots, [\alpha_m, l_m], F_k \rangle \quad (3.1)$$

where  $\alpha_1, \dots, \alpha_m$  are a choice of  $m$  meridians for  $L$ ,  $F_k$  is the  $k^{\text{th}}$  term of the lower central series of  $F = F\langle \alpha_1, \dots, \alpha_m \rangle$ , the free group on  $m$  generators and  $l_i$  is the  $i^{\text{th}}$  longitude of  $L$  written as a product of the  $\alpha_i$ 's.

With this presentation of  $G/G_k$ , the  $\bar{\mu}$ -invariants of a link  $L$  can be easily defined. Let  $\mathbb{Z}[[X_1, \dots, X_m]]$  be the ring of power series in  $m$  noncommuting variables. The *Magnus expansion, or embedding*, is a map  $E : \mathbb{Z}F \rightarrow \mathbb{Z}[[X_1, \dots, X_m]]$  defined by sending  $\alpha_i \mapsto 1 + X_i$  and  $\alpha_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 + \dots$  for  $1 \leq i \leq m$ . Let  $I = i_1 i_2 \dots i_{k-1} i_k$  be a string of integers amongst  $\{1, \dots, m\}$  with possible repeats. The Magnus expansion of the longitude  $l_{i_k}$  written as an element of  $F$  (modulo  $G_k$ ) has the form

$$E(l_{i_k}) = 1 + \sum \mu_L(i_1 \dots i_{k-1}) X_{i_1} \cdots X_{i_{k-1}}.$$

Milnor's invariant  $\bar{\mu}_L(I)$  is defined as the residue class of  $\mu_L(I)$  modulo the greatest common divisor of  $\mu_L(\tilde{I})$  where  $\tilde{I}$  is any string of integers obtained from  $I$  by deleting at least one integer (excluding  $i_k$ ) and cyclically permuting the rest. It is useful to note

that the first nonvanishing  $\bar{\mu}$ -invariant,  $\bar{\mu}_L(L)$  will be  $\mu_L(I)$  since it is well-defined.

For  $\bar{\mu}$ -invariants of length two, the calculation measures the linking between two components, i.e.  $\bar{\mu}_L(ij)$  is the linking number between the  $i^{\text{th}}$  and  $j^{\text{th}}$  components of  $L$ . It is also worth noting, even though we will not use this fact, that if all of the  $i_1, i_2, \dots, i_{k-1}, i_k$  are distinct, then  $\bar{\mu}$  is a link homotopy invariant [Mil54], [Mil57].

**Example 3.2.** Let  $BR = \text{Borromean Rings}$ , and  $\alpha_i$  be the meridian of  $L_i$ , the  $i^{\text{th}}$  component (see Figure 3.1), and  $l_i$  the respective longitude. Let  $G = \pi_1(S^3 - BR)$ . A presentation of  $G/G_k$  is given by

$$G/G_k \cong \langle \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, l_1], [\alpha_2, l_2], [\alpha_3, l_3], F_k \rangle$$

where  $l_1 = [\alpha_2, \alpha_3^{-1}]$ ,  $l_2 = [\alpha_3, \alpha_1^{-1}]$ , and  $l_3 = [\alpha_1, \alpha_2^{-1}]$ . In this example, the longitudes are independent of  $k$ .

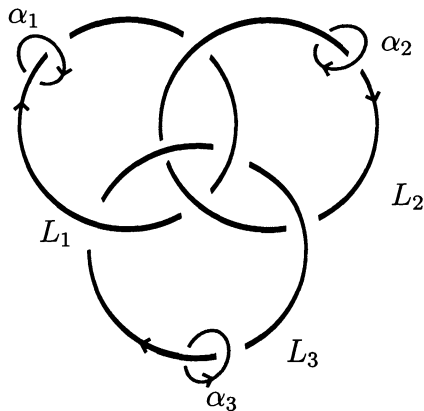


Figure 3.1: The Borromean Rings with meridians for each component

Now consider the Magnus expansion of  $l_3$ ,  $E(l_3)$ .

$$\begin{aligned}
 E(l_3) &= E(\alpha_1 \alpha_2^{-1} \alpha_1^{-1} \alpha_2) \\
 &= E(\alpha_1) E(\alpha_2^{-1}) E(\alpha_1^{-1}) E(\alpha_2) \\
 &= 1 - X_1 X_2 + X_2 X_1 + \text{higher order terms}
 \end{aligned}$$

The invariant  $\bar{\mu}_{BR}(123)$  will be the coefficient of  $X_1 X_2$ . Since there are no lower order terms besides 1, this is the first nonvanishing  $\bar{\mu}$ -invariant and  $\bar{\mu}_{BR}(123) = -1$  is well-defined. Also,  $\bar{\mu}_{BR}(213) = 1$ . Changing the orientation on a component will change the invariant by a sign.

## 3.2 Properties of Milnor's $\bar{\mu}$ -Invariants

An important property of  $\bar{\mu}$ -invariants is that they are concordance invariants [Cas75] and thus make Milnor's invariants a valuable invariant for us.

The following is a classical and well-known result of Milnor [Mil57].

**Theorem 3.3** (Milnor). *The longitudes of  $L$  lie in  $G_{k-1}$  if and only if  $F/F_k \cong G/G_k$ .*

*In other words,  $\bar{\mu}_L(I) = 0$  for  $|I| \leq k - 1$  if and only if  $F/F_k \cong G/G_k$ .*

The following corollary allows us to detect whether certain Milnor's invariants are zero using the fundamental group of  $M_L$ , the zero-framed surgery on  $L$ .

**Corollary 3.4.**  *$F/F_{k+1} \cong G/G_{k+1}$  if and only if  $F/F_k \cong J/J_k$ , where  $J = \pi_1(M_L)$ .*

*Proof.* The group  $G/G_k$  has presentation given by

$$G/G_{k+1} \cong \langle x_1, \dots, x_m | [x_i, \lambda_i], F_{k+1} \rangle$$

where  $\lambda_i$  is the longitude of the  $i^{\text{th}}$  component of  $L$ ,  $L_i$ , and  $x_i$  is a meridian of  $L_i$ . Consider zero-framed surgery on  $L$ . The inclusion of  $S^3 - L$  into  $M_L$  induces an epimorphism on fundamental groups that has kernel normally generated by  $\lambda_1, \dots, \lambda_m$ . The fundamental group  $J$  is obtained from  $G$  by setting the longitudes  $\lambda_i$  to zero. This gives the presentation  $J/J_k \cong \langle x_1, \dots, x_m | \lambda_i, F_{k+1} \rangle$ .

Suppose that the map induced from inclusion from  $G/G_{k+1}$  to  $F/F_{k+1}$  is an isomorphism. Then  $[x_i, \lambda_i] \in F_{k+1}$ , and thus  $\lambda_i \in F_k$  since  $x_i$  is a generator of  $F$ . Taking this information and looking at the presentation of  $J/J_k$  it is apparent that  $J/J_k \cong F/F_k$ .

Conversely, if  $J/J_k \cong F/F_k$  then the relations show that  $\lambda_i \in F_k$  and thus  $[x_i, \lambda_i] \in F_{k+1}$ . This gives that  $G/G_{k+1} \cong F/F_{k+1}$ .

□

**Remark 3.5.** *It follows that  $\bar{\mu}_L(I) = 0$  for  $|I| \leq k$  if and only if  $F/F_k \cong J/J_k$ .*

The following property of  $\bar{\mu}$ -invariants will be very useful in Chapter 5. The first nonvanishing  $\bar{\mu}$ -invariants are additive. This was first established by Orr [Orr89], but more geometric arguments can be found in [Coc90], [Ste90]. We will give a general idea of the proof for your convenience.

**Theorem 3.6 (Orr).** *Let  $L_0$  and  $L_1$  be in  $\mathcal{C}^m$ . If  $\bar{\mu}_{L_i}(I) = 0$  for all  $|I| \leq k$  and*

$\bar{\mu}_{\widehat{L}_i}(i_1 \dots i_{k+1}) = n_i$  for  $i = 0, 1$ , then  $\bar{\mu}_{\widehat{L_0 L_1}}(i_1 \dots i_{k+1}) = n_0 + n_1$ .

*Proof.* Let  $\widehat{L}_0$  and  $\widehat{L}_1$  be two  $m$ -component links in  $S^3$ . Let  $S$  be a 2-sphere in  $S^3$  such that  $S$  will separate  $S^3 - \widehat{L_0 L_1}$  into two components  $Y_0$  and  $Y_1$  with  $(S^3 - L_i) - \text{int}(Y_i)$  is the complement of a trivial string link (see Figure 3.2).

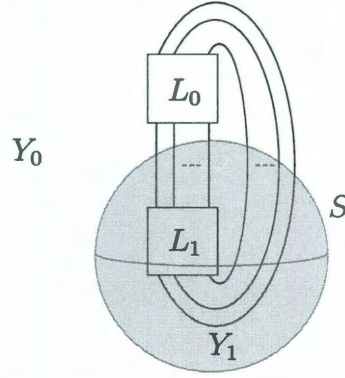


Figure 3.2: The sphere  $S$  separating  $S^3 - \widehat{L_0 L_1}$

Consider the following diagram of groups and maps.

$$\begin{array}{ccccc}
 & & \frac{\pi_1(S^3 - Y_0)}{\pi_1(S^3 - Y_0)_{k+1}} & \longrightarrow & \frac{\pi_1(S^3 - \widehat{L_0})}{\pi_1(S^3 - \widehat{L_0})_{k+1}} \\
 & \nearrow & \downarrow & & \\
 \frac{F}{F_{k+1}} & \xrightarrow{\quad} & \frac{\pi_1(S^3 - \widehat{L_0 L_1})}{\pi_1(S^3 - \widehat{L_0 L_1})_{k+1}} & & \\
 & \searrow & \uparrow & & \\
 & & \frac{\pi_1(S^3 - Y_1)}{\pi_1(S^3 - Y_1)_{k+1}} & \longrightarrow & \frac{\pi_1(S^3 - \widehat{L_1})}{\pi_1(S^3 - \widehat{L_1})_{k+1}}
 \end{array}$$

The left most isomorphisms are induced by the meridinal map. With some work, it can be shown that all the other maps are isomorphisms. The  $j^{\text{th}}$  longitude in  $L_i$ ,  $l_j^i$ , can be expressed as the same product of meridians in  $\pi_1(S^3 - \widehat{L_0 L_1})/\pi_1(S^3 - \widehat{L_0 L_1})_{k+1}$

as in  $\pi_1(S^3 - \widehat{L}_i)/\pi_1(S^3 - \widehat{L}_i)_{k+1}$  for  $i = 0, 1$ . Looking at the Magnus expansion, we have the following equality.

$$E(l_j^0 l_j^1) = 1 + \Sigma \mu_{\widehat{L_0 L_1}}(i_1 \dots i_s, j) X_{i_1} \dots X_{i_s}.$$

We can also consider the Magnus expansion of  $l_j^0 l_j^1$  as a product,  $E(l_j^0)E(l_j^1)$ .

$$\begin{aligned} E(l_j^0 l_j^1) &= E(l_j^0)E(l_j^1) \\ &= [1 + \Sigma \mu_{\widehat{L_0}}(\alpha_1 \dots \alpha_t, j) X_{\alpha_1} \dots X_{\alpha_t}] [1 + \Sigma \mu_{\widehat{L_1}}(\beta_1 \dots \beta_u, j) X_{\beta_1} \dots X_{\beta_u}] \end{aligned}$$

Combining the equalities gives the following

$$\begin{aligned} 1 + \Sigma \mu_{\widehat{L_0 L_1}}(i_1 \dots i_s, j) X_{i_1} \dots X_{i_s} &= \\ &= [1 + \Sigma \mu_{\widehat{L_0}}(\alpha_1 \dots \alpha_t, j) X_{\alpha_1} \dots X_{\alpha_t}] [1 + \Sigma \mu_{\widehat{L_1}}(\beta_1 \dots \beta_u, j) X_{\beta_1} \dots X_{\beta_u}] \end{aligned}$$

Comparing the coefficients, we see that for  $s \leq 2k - 1$ ,

$$\mu_{\widehat{L_0 L_1}}(i_1 \dots i_s, j) = \mu_{\widehat{L_0}}(\alpha_1 \dots \alpha_t, j) + \mu_{\widehat{L_1}}(\beta_1 \dots \beta_u, j).$$

Since  $\bar{\mu}$ -invariants of length  $k + 1$  for  $\widehat{L_0 L_1}$  have no indeterminacy,

$$\bar{\mu}_{\widehat{L_0 L_1}}(i_1 \dots i_k, j) = \bar{\mu}_{\widehat{L_0}}(i_1 \dots i_k, j) + \bar{\mu}_{\widehat{L_1}}(i_1 \dots i_k, j).$$

□

### 3.3 Relationship between $\bar{\mu}$ and $(n)$ -Solvability

Before now, little has been known about the relationship between Milnor's invariants and  $(n)$ -solvability. The following theorem demonstrates a relationship between the two concepts. Applications of this result will be seen in Chapter 5.

**Theorem 3.7.** *If  $L$  is an  $(n)$ -solvable link with  $m$  components, then  $\bar{\mu}_L(I) = 0$  for  $|I| \leq 2^{n+2} - 1$ .*

*Proof.* As mentioned in the previous section, a classical result of Milnor, given in Theorem 3.3 states that  $\bar{\mu}_L(I)=0$  for all  $|I| \leq k$  (for any link  $L$  in  $S^3$ ) if and only if  $F/F_{k+1} \cong G/G_{k+1}$ , where  $F = F\langle x_1, \dots, x_m \rangle$  and  $G = \pi_1(S^3 - L)$ . Using Lemma 3.4, this is equivalent to  $F/F_k$  being isomorphic to  $J/J_k$  where  $J = \pi_1(M_L)$ .

Suppose  $L$  is an  $(n)$ -solvable link with  $m$  components. By definition, there exists an  $(n)$ -solution,  $W$ , for  $L$ . Consider the following sequence of maps on  $\pi_1$  induced by inclusion (We are viewing  $F$  as the fundamental group of a wedge of  $m$  circles)

$$F \xrightarrow{\phi_1} G \xrightarrow{\phi_2} J \xrightarrow{\phi_3} E = \pi_1(W).$$

The map  $\phi_2$  is the surjection induced by the inclusion of  $S^3 - L$  into  $M_L$  and has kernel normally generated by the longitudes. The quotients of all of these groups by the  $k^{\text{th}}$  terms of their lower central series gives another sequence of maps

$$F/F_k \xrightarrow{\bar{\phi}_1} G/G_k \xrightarrow{\bar{\phi}_2} J/J_k \xrightarrow{\bar{\phi}_3} E/E_k.$$

Consider the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi_2} & J \\ l \downarrow & & \downarrow p \\ G/G_k & \xrightarrow{\overline{\phi_2}} & J/J_k \end{array}$$

where  $l$  and  $p$  are the canonical quotient maps. If  $[j] \in J/J_k$ , then, since  $\phi_2$  is a surjection, there exists  $g \in G$  such that  $\phi_2(g) = j$ . Now  $l(g) = [g]$  and thus  $\overline{\phi_2}([g]) = [j]$ . In turn, this gives that  $\overline{\phi_2} : G/G_k \rightarrow J/J_k$  is a surjection for all values of  $k$ .

To proceed, it is useful to know how to translate group theoretic results into results about topological spaces. If  $X$  is a connected complex then  $H_1(X; \mathbb{Z}) \cong H_1(\pi_1(X); \mathbb{Z})$  and  $H_2(\pi_1(X); \mathbb{Z})$  is a quotient of  $H_2(X; \mathbb{Z})$ . In fact, the Hurewicz map induces an exact sequence

$$\pi_2(X) \rightarrow H_2(X) \rightarrow H_2(\pi_1(X)) \rightarrow 1. \quad (3.2)$$

Dwyer's Theorem [Dwy75] is of particular importance and is stated below.

**Theorem 3.8** (Dwyer's Integral Theorem). *Let  $\phi : A \rightarrow B$  be a homomorphism that induces an isomorphism on  $H_1(-; \mathbb{Z})$ . Then for any positive integer  $k$ , the following are equivalent:*

- i.  $\phi$  induces an isomorphism  $A/A_{k+1} \cong B/B_{k+1}$*
- ii.  $\phi$  induces an epimorphism  $H_2(A; \mathbb{Z})/\Phi_k(A) \rightarrow H_2(B; \mathbb{Z})/\Phi_k(B)$ .*

where  $\Phi_k(A) = \ker(H_2(A) \rightarrow H_2(A/A_k))$  for  $k \geq 1$ .



Consider the map induced by  $\phi_3 \circ \phi_2 \circ \phi_1$

$$H_2(F; \mathbb{Z}) / \Phi_k(F) \rightarrow H_2(E; \mathbb{Z}) / \Phi_k(E) \quad (3.3)$$

where  $E = \pi_1(W)$ . Thus showing (3.3) is a surjection is equivalent to showing  $\phi := \overline{\phi_3} \circ \overline{\phi_2} \circ \overline{\phi_1} : F/F_{k+1} \rightarrow E/E_{k+1}$  being an isomorphism.

Since  $F$  is the free group on  $m$  generators,  $H_2(F; \mathbb{Z}) = 0$ . The map of (3.3) is a surjection precisely when  $\Phi_k(E) = H_2(E, \mathbb{Z})$ . We need to determine for which  $k$  we have  $\Phi_k(E) = H_2(E, \mathbb{Z})$ .

Since  $W$  is an  $(n)$ -solution for  $L$ , there is a basis of  $H_2(W)$  consists of pairs of surfaces,  $\{L_i, D_i\}_{i=1}^r$  such that  $L_i \cap D_j = \delta_{i,j}$ . By the exact sequence 3.2,  $H_2(W) \rightarrow H_2(E)$  is a surjection and is induced by the inclusion map. Thus  $H_2(E)$  is generated by the images of the  $L_i$  and  $D_i$ .

A reformulation of  $\Phi_k(E)$  given by Cochran and Harvey [CH08] is of use and will be stated here. For any space  $X$ ,  $\Phi_k(X)$  is the subgroup of  $H_2(X)$  consisting of those elements that can be represented by an oriented surface  $f : \Sigma \rightarrow X$  such that for some symplectic basis of curves  $\{a_i, b_i | 1 \leq i \leq \text{genus}(\Sigma)\}$  of  $\Sigma$ ,  $f_*([a_i]) \subset \pi_1(X)_k$ . In other words, one half of a symplectic basis of curves map into  $\pi_1(X)_k$ . Note that  $\Phi_n(E)$  is the same as  $\Phi_n(K(E, 1))$  in the sense of this reformulation. Recall that a space  $X$  is  $K(G, n)$  if  $\pi_n(X) \cong G$  and  $\pi_i(X) = 0$  for  $i \neq n$ .

We consider  $\Phi_k(E)$  in terms of this reformulation. We know that  $H_2(E)$  is generated by the images of  $L_i$  and  $D_i$ . A symplectic basis for each of these surfaces lie in  $\pi_1(W)^{(n)}$  since  $\pi_1(L_i), \pi_1(D_i) \in \pi_1(W)^{(n)}$ . We recall that the derived series and

lower central series are related by  $G^{(n)} \subseteq G_{2^n}$ . Thus, every element of  $H_2(E)$  can be represented by oriented surfaces with basis curves in  $E_{2^n}$ . Hence  $\Phi_k(E) = H_2(E)$  for  $k \leq 2^n$ .

Using Dwyer's Theorem 3.8, we have that  $\phi_3 \circ \phi_2 \circ \phi_1$  induces an epimorphism

$$H_2(F)/\Phi_k(F) \rightarrow H_2(E)/\Phi_k(E)$$

for  $k \leq 2^n$  and in turn gives an isomorphism

$$F/F_{k+1} \rightarrow E/E_{k+1}$$

for  $k \leq 2^n$ .

Thus  $\hat{\phi} := \overline{\phi_2} \circ \overline{\phi_1} : F/F_{2^{n+1}} \rightarrow J/J_{2^{n+1}}$  is a monomorphism. Since  $\hat{\phi}$  is a map  $F/F_k \rightarrow F/\langle \text{relations}, F_k \rangle$ , by Milnor's presentation (3.1),  $\hat{\phi}$  is a surjection and thus an isomorphism.

By Lemmas 3.3 and 3.4, the  $\bar{\mu}$ -invariants of length less than or equal to  $2^n + 1$  vanish for  $(n)$ -solvable links.

We can better this result. By considering the following diagram.

$$\begin{array}{ccccc} H_2(W_k) & \xrightarrow{p_*} & H_2(W) & & \\ \downarrow & & \downarrow & & \\ H_2(E_k) & \xrightarrow{i_*} & H_2(E) & \xrightarrow{\pi_*} & H_2(E/E_{2^{k-1}}) \end{array}$$

where  $W_k$  is the covering space of  $W$  that corresponds with the  $k^{\text{th}}$  term of the

lower central series of  $\pi_1(W)$ . The vertical maps are surjections obtained from the exact sequence induced by the Hurewicz map. The maps  $p_*$ ,  $i_*$  and  $\pi_*$  are the maps induced by the covering map  $p$ , inclusion and projection respectively.

The images of the basis  $\{L_i, D_i\}$  of  $H_2(W)$  will generate  $H_2(E)$  since  $H_2(W) \rightarrow H_2(E)$  is a surjection. We claim that the map  $i_*$  is a surjection. This can be seen by viewing  $H_2(E_k)$  as the second homology group for the covering space,  $K(E_k, 1)$  of the Eilenberg-MacLane space  $K(E, 1)$ . Note that  $K(E_k, 1)$  is the covering space of  $K(E, 1)$  corresponding to the subgroup  $E_k$  of  $E$ . When  $k = 2^n$  the images of  $\{L_i, D_i\}$  in  $H_2(E)$  will lift to  $H_2(E_k)$  so  $i_* : H_2(E_k) \rightarrow H_2(E)$  is surjective.

Cochran and Harvey [CH10] showed that the composition of the following maps

$$H_2(E_k) \xrightarrow{i_*} H_2(E) \xrightarrow{\pi_*} H_2(E/E_{2k-1})$$

is the zero map for all  $k$ . Since  $i_*$  is surjective, this implies that  $\pi_*$  is the zero map. Hence  $\Phi_{2k-1}(E) = H_2(E)$  and Dwyer's theorem gives an isomorphism  $F/F_{2k} \cong E/E_{2k}$  and thus an isomorphism  $F/F_{2k} \cong J/J_{2k}$  when  $k = 2^n$  using a similar argument as above. Therefore the  $\bar{\mu}$ -invariants of length less than or equal to  $2^{n+1}$  vanish.

This result can be improved slightly. Let  $g : J/J_{2^{n+1}} \rightarrow F/F_{2^{n+1}}$  be a specified isomorphism. Let  $f$  be the composite of the following maps

$$J \xrightarrow{\pi_J} J/J_{2^{n+1}} \xrightarrow{g} F/F_{2^{n+1}}$$

where  $\pi_J$  is the canonical quotient map. Consider the following diagram of maps

$$\begin{array}{ccccc}
 E/E_{2^{n+1}} & \xleftarrow[\cong]{\phi} & J/J_{2^{n+1}} & \xleftarrow[\cong]{g^{-1}} & F/F_{2^{n+1}} \\
 \uparrow \pi_E & & \uparrow \pi_J & \nearrow g \circ \pi_J & \\
 E & & J & & 
 \end{array}$$

where  $\phi$  is the isomorphism between  $J/J_{2^{n+1}}$  and  $E/E_{2^{n+1}}$  established earlier in the proof and  $\pi_E$  is the canonical quotient map. Thus we have an extension of  $f$ , namely  $\bar{f} = g \circ \phi^{-1} \circ \pi_E : E \rightarrow F/F_{2^{n+1}}$ . This gives the following commutative diagram.

$$\begin{array}{ccc}
 \pi_1(M_L) & \xrightarrow{f} & F/F_{2^{n+1}} \\
 \downarrow i_* & \nearrow \bar{f} & \\
 \pi_1(W) & & 
 \end{array}$$

The commutative diagram below on homology is achieved by the induced maps obtained from the above maps.

$$\begin{array}{ccc}
 H_3(M_L) & \xrightarrow{f} & H_3(F/F_{2^{n+1}}) \\
 \downarrow i_* & \nearrow \bar{f} & \\
 H_3(W) & & 
 \end{array}$$

The images of  $H_3(M_L)$  in  $H_3(W)$  will be zero since  $\partial W = M_L$ , and since the diagram commutes, the image of  $H_3(M_L)$  in  $H_3(F/F_{2^{n+1}})$  will be zero. In other words,  $[M_L] \rightarrow 0 \in H_3(F/F_{2^{n+1}})$ . Consider the following sequence of maps

$$H_3(F/F_{2^{k-1}}) \rightarrow H_3(F/F_{2^{k-2}}) \rightarrow \cdots \rightarrow H_3(F/F_{k+2}) \rightarrow H_3(F/F_{k+1}) \rightarrow H_3(F/F_k)$$

where  $h_i : H_3(F/F_{k+i}) \rightarrow H_3(F/F_{k+i-1})$  and  $k = 2^{n+1}$ . The image of the fundamental class under the map  $H_3(M_L) \rightarrow H_3(F/F_m)$  will be denoted by  $\theta_m(M_L, f)$ .

We will use the following two results of Cochran, Gerges and Orr [CGO01]. They will be stated without proof. It is worth noting that these results rely heavily on deep work of Igusa and Orr [IO01].

**Lemma 3.9** (Cochran-Gerges-Orr).  $\theta_m(M_L, f) \in \text{Image}(\pi_* : H_3(F/F_{m+1}) \rightarrow H_3(F/F_m))$  if and only if there is some isomorphism  $\tilde{f} : J/J_{m+1} \rightarrow F/F_{m+1}$  extending  $f$  such that  $\pi_*(\theta_{m+1}(M_L, \tilde{f})) = \theta_m(M_L, f)$ .

**Corollary 3.10** (Cochran-Gerges-Orr). *The map  $H_3(F/F_{2m-1}) \rightarrow H_3(F/F_m)$  is the zero map. Any element in the kernel of  $H_3(F/F_{m+j}) \rightarrow H_3(F/F_m)$ ,  $j \leq m-1$ , lies in the image of  $H_3(F/F_{2m-1}) \rightarrow H_3(F/F_{m+j})$ .*

Since  $\theta_{2^{n+1}}(M_L, f) = [0]$  and  $[0]$  is always in the image of a homomorphism, there is an extension of  $f$  to an isomorphism  $\tilde{f} : J/J_{k+1} \rightarrow F/F_{k+1}$  with  $h_1(\theta_{k+1}(M_L, \tilde{f})) = \theta_k(M_L, f) = 0$  by Lemma 3.9. This means that  $\theta_{k+1}(M_L, \tilde{f})$  is in the kernel of  $h_1$ . So  $\theta_{k+1}(M_L, \tilde{f})$  lies in the image of  $H_3(F/F_{2k-1}) \rightarrow H_3(F/F_{k+1})$  by Corollary 3.10. In other words, it lies in the image of the map  $h_2 \circ h_3 \circ \dots \circ h_{k-1}$  and in turn lies in the image of  $h_2$ . By Lemma 3.9, there is an extension of  $\tilde{f}$  that is an isomorphism between  $J/J_{k+2}$  and  $F/F_{k+2}$ . By continuing this process, an isomorphism between  $J/J_{2k-1}$  and  $F/F_{2k-1}$  with  $k = 2^{n+1}$  is obtained. Thus we have that the  $\bar{\mu}$ -invariants of lengths less than or equal to  $2^{n+2} - 1$  of our  $(n)$ -solvable link vanish. This concludes the proof of Theorem 3.7. □

**Example 3.11.** Consider the Borromean Rings =  $BR$ . In Example 3.1 it was shown

that  $\bar{\mu}_{BR}(123) = \pm 1$ , depending on the orientation of  $BR$ . As Theorem 3.7 states, all  $\bar{\mu}$ -invariants vanish for lengths less than or equal to three for (0)-solvable links. Thus  $BR$  cannot be (0)-solvable, and  $BR$  is nontrivial in  $\mathcal{F}_{-0.5}^3/\mathcal{F}_0^3$ .

**Remark 3.12.** *The converse of Theorem 3.7 is false. Consider the Whitehead link  $= W$  in Figure 3.3. The first nonvanishing  $\bar{\mu}$ -invariant occurs at length four. One of these invariants is  $\bar{\mu}_W(1122) = \pm 1$ , depending on orientation. To see this, note that the figure eight knot,  $4_1$ , maybe obtained as the result of band summing the two components of  $W$ . It is known that this knot is not (0)-solvable since its Arf invariant is nonzero [COT03]. By Proposition 2.10, the Whitehead link is not (0)-solvable.*

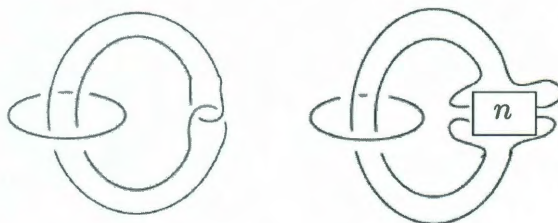


Figure 3.3: The Whitehead Link and the  $n$ -twisted Whitehead link

**Remark 3.13.** *The result of Theorem 3.7 is sharp in the sense that we cannot increase the length of vanishing  $\bar{\mu}$ -invariants. Consider the  $n$ -twisted Whitehead link in Figure 3.3. The number  $n$  represents the number of full twists. When  $n$  is even, this link is band pass equivalent to the trivial link. As we will see in Chapter 4, this link is (0)-solvable since the trivial link is (0)-solvable. However,  $\bar{\mu}(1122) = -n$  and  $\bar{\mu}(1212) = 2n$  are the first nonvanishing  $\bar{\mu}$ -invariants.*

# Chapter 4

## Bing Doubling

### 4.1 Introduction

The goal of this chapter is to investigate the effect of Bing doubling on  $(n)$ -solvable links. We begin with the definition of this operation. Bing doubling is a doubling operator performed on knots and links. If  $K$  is a knot, then the two-component link in Figure 4.1 is the Bing double of  $K$ , denoted  $BD(K)$ . If  $L$  is an  $m$ -component link,  $BD(L)$  denotes be the  $2m$ -component link obtained by Bing doubling every component of  $L$ .

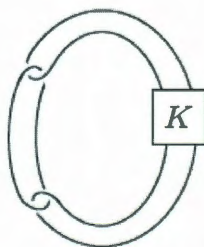


Figure 4.1: The Bing double of a knot  $K$ ,  $BD(K)$

Most constructions of Bing doubling involve a type of satellite construction called

genetic infection. This construction is as follows. Let  $M = S^3 - N(L)$  where  $L$  is an  $m$ -component link in  $S^3$  and  $\eta$  is a curve in  $S^3 - N(L)$ . Let  $K$  be a knot in  $S^3$ . Then

$$M(\eta, K) := (M - N(\eta)) \cup_f (S^3 - N(K))$$

where  $f : \partial(S^3 - N(K)) \rightarrow \partial(M - N(\eta))$  is defined by  $f_*(\mu_K) = l_\eta^{-1}$  and  $f_*(l_K) = \mu_\eta$ . Note that  $\mu_K$  is the meridian of  $\partial(N(K))$  and  $l_K$  is the longitude of  $\partial(N(K))$  and similar definitions for  $\mu_\eta$  and  $l_\eta$ .

The manifold  $M(\eta, K)$  is homeomorphic to  $S^3 - N(L(\eta, K))$  where  $L(\eta, K)$  is another  $m$ -component link. We say  $L(\eta, K)$  is the result of infecting  $L$  along  $\eta$  by  $K$ . Milnor's  $\bar{\mu}$ -invariants are, however, unchanging under this construction. The following lemma will be of use in the proof.

**Lemma 4.1.** *If  $f : X \rightarrow Y$  and  $f_* : H_2(X) \rightarrow H_2(Y)$  is surjective, then there is an epimorphism induced by  $f_*$  on  $H_2(\pi_1(X)) \rightarrow H_2(\pi_1(Y))$ .*

*Proof.* We will prove the special case when  $X = M(\eta, K)$  and  $Y = M$ . The general case of this lemma, with arbitrary spaces  $X$  and  $Y$  can be proven in a similar manner.

Let  $A = \pi_1(M(\eta, K))$  and  $B = \pi_1(M)$  Consider the following diagram of maps.

$$\begin{array}{ccc} M & \longrightarrow & K(B, 1) \\ f \uparrow & & \uparrow h \\ M(\eta, K) & \longrightarrow & K(A, 1) \end{array}$$

Notice that  $K(A, 1) = M(\eta, K) \cup 3\text{-cells} \cup 4\text{-cells} \cup \dots$  and  $K(B, 1) = M \cup 3\text{-cells} \cup 4\text{-cells} \cup \dots$ . The map  $f$  can be extended to  $h$  to give a commutative diagram,



where  $h|_{M(\eta, K)}$  is precisely the map  $f$ , since  $\pi_i(K(B, 1)) = 1$  for  $n \geq 2$ . Taking the diagram from above and applying the hom functor gives the following commutative diagram.

$$\begin{array}{ccc} H_n(M) & \longrightarrow & H_n(K(B, 1)) \\ f_* \uparrow & & \uparrow h_* \\ H_n(M(\eta, K)) & \longrightarrow & H_n(K(A, 1)) \end{array}$$

Since only 3 dimensional and higher cells are being added to  $M$  to obtain  $K(B, 1)$ , we have that the map  $H_n(M) \rightarrow H_n(K(B, 1))$  is surjective for  $n = 1, 2$ . Similarly,  $H_n(M(\eta, K)) \rightarrow H_n(K(A, 1))$  is surjective for  $n = 1, 2$ . Since all other maps in the diagram are surjective and the diagram commutes,  $h_*$  is also surjective.

□

**Proposition 4.2.** *Milnor's  $\bar{\mu}$ -invariants are unchanged by infection by a knot along a curve.*

*Proof.* For any knot  $K$  in  $S^3$ , there exists a map  $h : S^3 - K \rightarrow S^3 - U$  where  $U$  is the unknot and the map fixes the boundary. This map will induce an isomorphism on homology. This in turn gives a map  $\bar{h} : M(\eta, K) \rightarrow M$  that induces an isomorphism on homology. We can see by Lemma 4.1, there is an epimorphism induced by  $\bar{h}_*$  on  $H_2(\pi_1(M(\eta, K)))$  to  $H_2(\pi_1(M))$ . We can now apply Stallings' Integral Theorem [Sta65].

**Theorem 4.3** (Stallings' Integral Theorem). *Let  $\phi : A \rightarrow B$  be a homomorphism that induces an isomorphism on  $H_1(-; \mathbb{Z})$  and an epimorphism on  $H_2(-; \mathbb{Z})$ . Then, for each  $n$ ,  $\phi$  induces an isomorphism  $A/A_k \cong B/B_k$ .*

We have that  $\pi_1(M(\eta, K))/\pi_1(M(\eta, K))_k \cong \pi_1(M)/\pi_1(M)_k$  for all  $k$ . This means precisely that the two links,  $L$  and  $L(\eta, K)$  have the same  $\bar{\mu}$ -invariants.  $\square$

## 4.2 Construction

Bing doubling can also be viewed as multi-infection by a string link. Let  $L = L_1 \cup L_2 \cup \cdots \cup L_m$  in  $S^3$  be an  $m$  component link in  $S^3$ . Let  $L_{BD}$  be the  $2m$ -component link pictured in Figure 4.2 that is isotopic to the  $2m$ -component trivial link. Then there is a handlebody,  $H$ , in  $S^3 - L_{BD}$  which is the exterior of a trivial string link with  $m$  components (see Figure 4.3 for an example). The  $\eta_i$  are curves in  $S^3 - L_{BD}$  and are the canonical meridians of the trivial string link.

Take a string link  $J$ , such that  $\hat{J}$  is isotopic to  $L$ . Recall that in doing so, our choice of  $d$ -base is not unique so there will be an infinite number of string links that meet this criterion. Then

$$BD(L) = ((S^3 - L_{BD}) - H) \cup_{\phi} (D^2 \times I - J)$$

where  $\phi$  maps  $l_i \mapsto \gamma_i$  and  $\mu_i \mapsto \eta_i^{-1}$  and the  $l_i$ ,  $\gamma_i$ ,  $\mu_i$  and  $\eta_i$  are depicted in Figure 4.4.

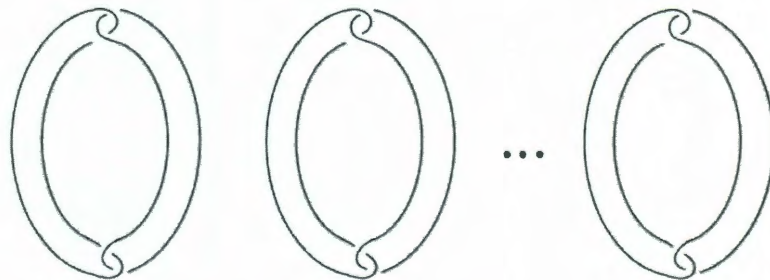
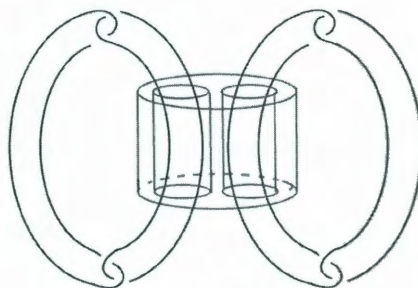
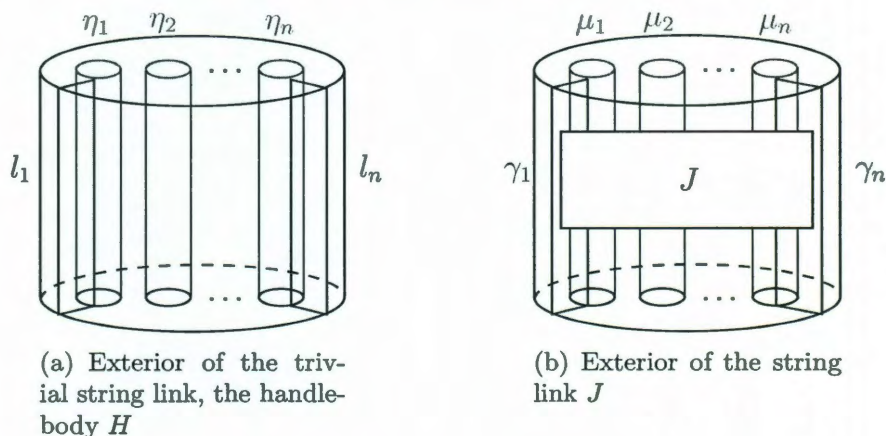


Figure 4.2: The trivial link  $L_{BD}$

Figure 4.3: A handlebody in  $S^3 - L_{BD}$ Figure 4.4: The longitudes  $l_i, \gamma_i$  and meridians  $\mu_i, \eta_i$ 

To help illustrate this construction, consider the following example.

**Example 4.4.** Let  $H$  be the Hopf link. The string link  $J$  (see Figure 4.5(a) for the complement of  $J$ ) is one in which when closed it is isotopic to  $H$ . The resulting link obtained after replacing the handlebody with the exterior of  $J$  is  $BD(H)$ .

### 4.3 Effects of Bing Doubling on $(n)$ -Solvability

The goal of this section is to understand the effect that Bing doubling will have on solvability. In order to do this, we first look at other geometric moves similar to the idea of Reidemeister moves that can be performed on knots and links and see their

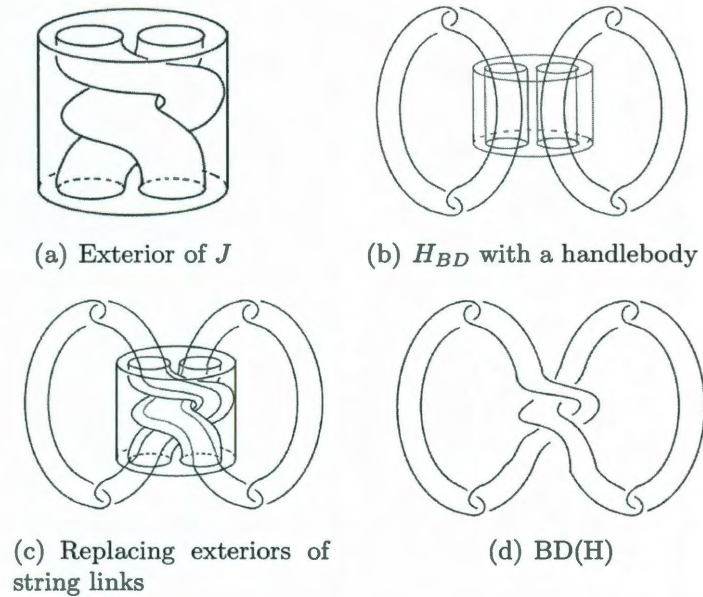


Figure 4.5: Illustration of the Bing Doubling of the Hopf link using infection by a string link

effects on solvability.

**Remark 4.5.** *Lemmas 4.6 and 4.9 are results of Taylor Martin [Mar]. At the current time, these results and proofs are not in print. We will state the results and give an idea of both proofs.*

**Lemma 4.6** (Martin). *A band pass move preserves (0)-solvability.*

*Proof.* Suppose that  $L$  is (0)-solvable and is band pass equivalent to  $J$ . Suppose further that  $L$  is obtained from  $J$  by one band pass move and  $L$  locally has Position A in its diagram (see Figure 4.6).

We will create a cobordism between  $M_L$  and  $M_J$ . We add two 2-handles,  $D_1, D_2$  to  $M_L \times I$  along the curves  $\gamma_1$  and  $\gamma_2$  with framing 0. Call the resulting space  $X$ , which will be our cobordism. Then  $\partial X = M_L \cup M_J$  since performing zero-framed surgery to  $\gamma_i$  and  $\gamma_2$  will give the link in Figure 4.6 (b). This can be seen by simple

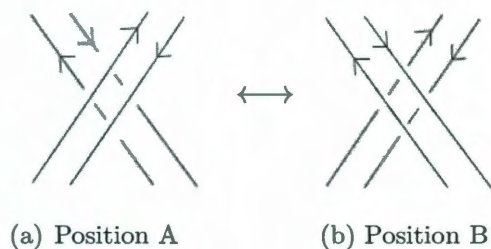


Figure 4.6: Band pass move

Figure 4.7: The addition of two components,  $\gamma_1$  and  $\gamma_2$ , that when we take the zero surgery we obtain Position B in Figure 4.6

handle slides.

Let  $S = B^4 - \mathbb{D}$  be a slice disk complement where  $\mathbb{D}$  are disks with boundary  $L$ . Now consider the manifold  $X \cup S$ . Since the curves  $\gamma_1$  and  $\gamma_2$  (again, see Figure 4.7) are of finite order,  $H_1(X \cup S) \cong H_1(S)$ . Adding the handles  $D_1$  and  $D_2$  creates a pair of surfaces  $\Sigma_1$  and  $\Sigma_2$  (see Figure 4.8 for  $\Sigma_1$ ) to the basis of  $H_2(S)$  to create a basis for  $H_2(X \cup S)$ , namely  $\{L_i, D_i\}_{i=1}^r \cup \{\Sigma_1, \Sigma_2\}$ . The third condition of (0)-solvability is vacuous since  $G^{(0)} = G$ . Thus (0)-solvability is preserved under band pass moves.

□

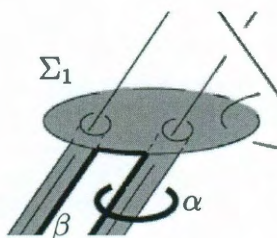


Figure 4.8: Zero surgery with added link components and the surfaces that are added as a result of adding the link components

**Proposition 4.7.** *If  $L$  is any link of  $m$  components, then  $BD(L)$  is (0)-solvable.*

*Proof.* Let  $L$  be an  $m$ -component link in  $S^3$ . The Bing double,  $BD(L)$  is band pass equivalent to the trivial link of  $m$  components (arising from the fact that any link can be transformed into the trivial link by a finite number of crossing changes). Since the trivial link is (0)-solvable and band pass moves preserve (0)-solvability,  $BD(L)$  is (0)-solvable.  $\square$

We will also consider two other geometric moves.



Figure 4.9: The delta move and the half-clasp move

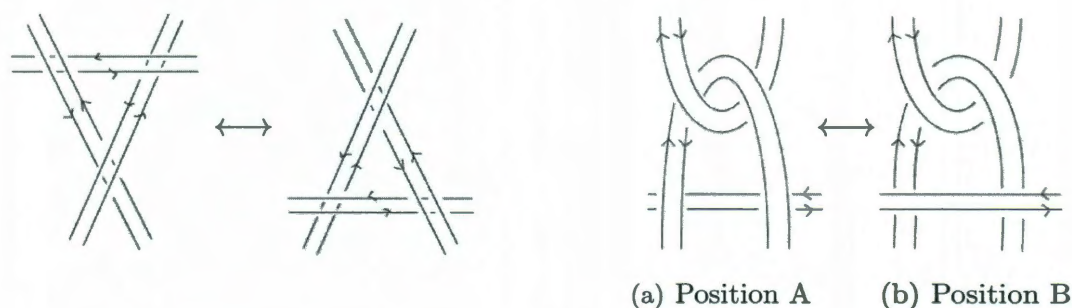


Figure 4.10: The double delta move and the double half-clasp move

**Lemma 4.8.** *The delta move can be realized as a half-clasp move. Moreover, the double delta move can be realized by a double half-clasp move.*

*Proof.* The images in Figure 4.11 illustrate how to use isotopy and a half-clasp move to achieve the delta move. This result is easily adaptable for the double of the moves.

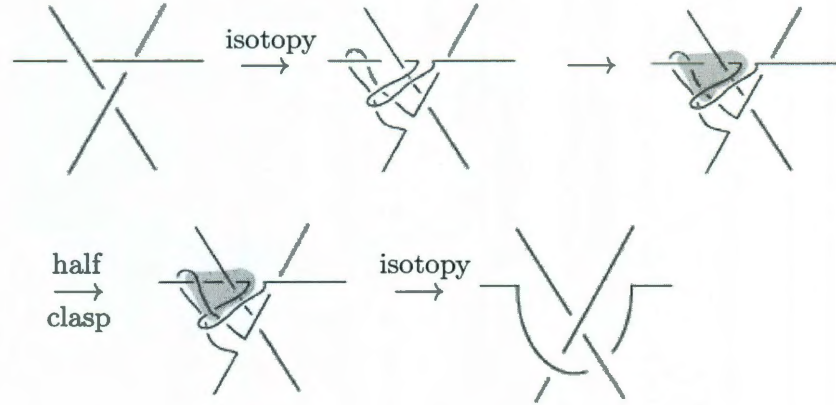


Figure 4.11: Sequence of isotopy and half-clasp moves to achieve the delta move.

□

**Lemma 4.9** (Martin). *The double half-clasp move preserves (0.5)-solvability.*

*Proof.* Suppose  $L$  and  $J$  are related by a double half-clasp move depicted in Figure 4.10. Moreover, suppose  $L$  is (0.5)-solvable with a (0.5)-solution  $W$ . Assume further that  $L$  has Position B somewhere in its diagram.

As in the proof of Lemma 4.6, we will create a cobordism between  $M_L$  and  $M_J$  by adding two 2-handles  $D_1$  and  $D_2$  to  $M_L \times I$  along the curves  $\gamma_1$  and  $\gamma_2$  with zero framing. Call the resulting cobordism  $X$ . Then  $\partial X = M_L \cup M_J$  since zero-framed surgery on  $\gamma_1$  and  $\gamma_2$  gives the link in Position A of Figure 4.10(a), which can be seen by handle slides.

We focus on the third criteria for (0.5)-solvability. Adding  $D_1$  and  $D_2$  creates the pair of surfaces  $\Sigma_1$  and  $\Sigma_2$  as seen in Figure 4.12 that will be added basis elements of  $H_2(W \cup X)$ . The curves  $\alpha$  and  $\beta$  in  $\Sigma_1$  are null homologous in  $M_L$ , i.e.  $\alpha, \beta \in [0] \in H_1(M_L)$ . Since the inclusion  $i : M_L \rightarrow W$  induces an isomorphism in homology,  $\alpha, \beta$  are zero in  $H_1(W)$ .  $\pi_1(\Sigma_1) \subset \pi_1(W)^{(1)} \hookrightarrow \pi_1(W \cup X)^{(1)}$ . By the same argument in

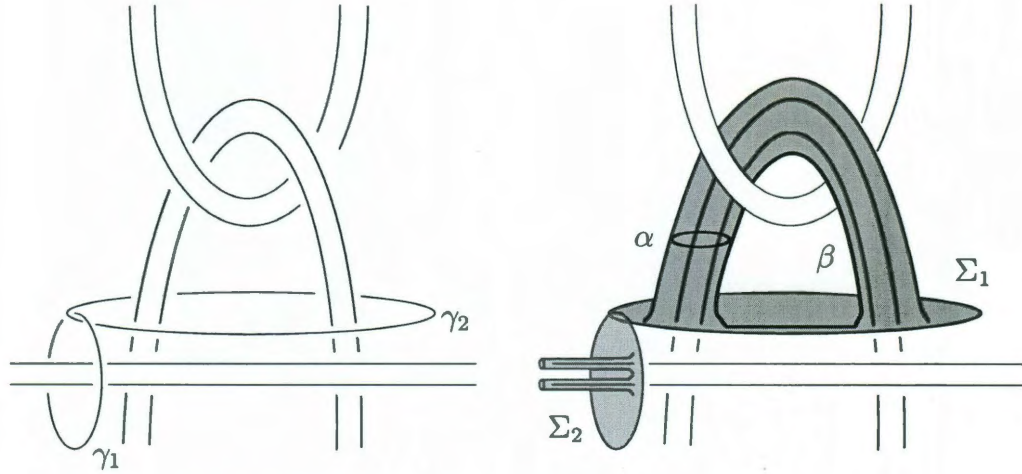


Figure 4.12: Zero surgery with added link components and the surfaces that are added as a result of adding the link components

Lemma 4.6, this proves that the double half-clasp moves preserve (0.5)-solvability.

□

**Proposition 4.10.** *If  $L \in \mathcal{F}_{-0.5}^m$ , then  $BD(L)$  is (0.5)-solvable.*

*Proof.* Suppose  $L$  has all pairwise linking numbers equal to zero. It was shown ([MN89], and [Mat87]) that two links are equivalent by delta moves if and only if they have the same pairwise linking numbers. This result was generalized for string links [NS03]. Recall that in our construction of Bing doubling of a link, we chose a string link  $J$  such that  $\hat{J}$  is isotopic to  $L$ . Since  $\hat{J}$  has all pairwise linking numbers equal to zero by assumption,  $J$  can be chosen to have all pairwise linking numbers equal to zero as a string link.

In the construction of Bing doubling we can see that the handlebody  $H$  was replaced with the exterior of  $J$ . As a result of this replacement, we have a new string link  $\tilde{J}$  (see Figure 4.5(c) for example). Using double delta moves, we are able to get the trivial link (delta moves on  $J$  will be double delta moves on  $\tilde{J}$ ). Since the double



half-clasp move preserves (0.5)-solvability, the double delta move will also preserve (0.5)-solvability. Thus  $BD(L)$  is (0.5)-solvable.  $\square$

**Proposition 4.11.** *If  $L$  is an  $(n)$ -solvable link, then  $BD(L)$  is  $(n + 1)$ -solvable.*

*Moreover, if  $L$  is an  $(n.5)$ -solvable link, then  $BD(L)$  is  $((n + 1).5)$ -solvable.*

*Proof.* Suppose  $L$  is an  $(n)$ -solvable link with  $m$  components. From the beginning of this section, we can construct  $BD(L)$  by infection by a string link on a trivial link. We will construct an  $(n + 1)$ -solution for  $BD(L)$  and begin by finding a cobordism between  $M_L$  and  $M_{BD(L)}$ . Suppose  $J$  is a string link such that  $\hat{J}$  is isotopic to  $L$ . Then  $M_L = (D^2 \times I - J) \cup (D^2 \times I - \text{trivial string link})$ . Consider  $M_L \times [0, 1]$  and  $M_{L_{BD}} \times [0, 1]$ . Recall that  $L_{BD}$  was isotopic to the  $2m$  component trivial link. Let  $H$  be the handlebody  $D^2 \times I$ . Glue  $M_L \times \{1\}$  to  $M_{L_{BD}} \times \{1\}$  by identifying  $H \subset M_L \times \{1\}$  with  $H \subset M_{L_{BD}} \times \{1\}$ . Call the resulting space  $X$  (see Figure 4.13). Then  $\partial X = M_L \amalg M_{L_{BD}} \amalg -M_{BD(L)}$ .

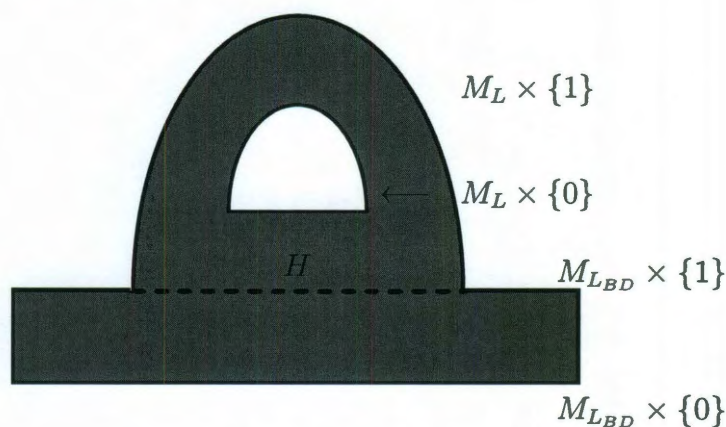


Figure 4.13: The space  $X$

To proceed, we need the following lemma.

**Lemma 4.12.** *With  $X$  as above, the inclusion maps induce the following*

- i. isomorphisms  $H_1(M_{L_{BD}}) \rightarrow H_1(X)$  and  $H_1(M_{BD(L)}) \rightarrow H_1(X)$ ;
- ii. isomorphism  $H_2(X) \cong H_2(M_{L_{BD}}) \oplus H_2(M_L)$ .

*Proof.* Consider the following diagram of inclusion maps.

$$\begin{array}{ccc}
 & H & \\
 i_1 \swarrow & & \searrow i_2 \\
 M_{L_{BD}} \times [0, 1] & & M_L \times [0, 1] \\
 j_2 \searrow & & \swarrow j_1 \\
 & X &
 \end{array}$$

Using Mayer Vietoris, the maps above induce the following long exact sequence (in reduced homology), where  $I_* = (i_{1*}, i_{2*})$  and  $J_* = j_{1*} - j_{2*}$  (the homology groups are with  $\mathbb{Z}$  coefficients).

$$\begin{aligned}
 \cdots & \xrightarrow{\partial_*} H_2(H) \xrightarrow{I_*} H_2(M_{L_{BD}}) \oplus H_2(M_L) \xrightarrow{J_*} H_2(X) \xrightarrow{\partial_*} \\
 & H_1(H) \xrightarrow{I_*} H_1(M_{L_{BD}}) \oplus H_1(M_L) \xrightarrow{J_*} H_1(X) \xrightarrow{\partial_*} 0.
 \end{aligned}$$

The homology group  $H_1(H) \cong \mathbb{Z}^m$  is generated by the meridians,  $\mu_i$  of the trivial string link. Recall that the  $\eta_i$ 's were defined in the construction of Bing doubling. Now  $i_{1*}(\mu_i) = 0$  in  $S^3 - L_{BD} \subset M_{L_{BD}}$  since  $\mu_i \sim \eta_i$  and  $\eta_i$  is in a commutator subgroup. Also,  $i_{2*}(\mu_i)$  is of infinite order in  $S^3 - L \subset M_L$  since  $\mu_i$  is identified with a meridian of  $L$ . Hence  $I_*$  is a monomorphism. Thus the map  $\partial_* : H_2(X) \rightarrow H_1(H)$

is the zero map. By the properties of a long exact sequence,

$$H_2(M_{L_{BD}}) \oplus H_2(M_L) \cong H_2(X).$$

For the other part of the lemma, consider the first isomorphism theorem. This gives

$$H_1(X) \cong \frac{H_1(M_{L_{BD}}) \oplus H_1(M_L)}{\text{image}(I_* : H_1(\eta \times D^2) \rightarrow H_1(M_{L_{BD}}) \oplus H_1(M_L))}.$$

The image of  $I_*$  is precisely  $H_1(M_L)$ . Thus  $H_1(X) \cong H_1(M_{L_{BD}}) \cong \mathbb{Z}^{2m}$ . Now  $H_1(M_{BD(L)})$  is generated by the meridians of  $BD(L)$  which are isotopic (in  $X$ ) to the meridians of  $L_{BD}$ . This means that  $H_1(X) \cong H_1(M_{BD(L)})$  which is the desired result.

□

We now continue with the proof of the proposition. Let  $S = B^4 - \mathbb{D}$  be a slice disk complement where  $\mathbb{D} \subset B^4$  is a collection of disks with boundary  $L_{BD}$ . Let  $W$  be an  $(n)$ -solution for  $L$  and let  $E$  be the space obtained by attaching  $W$  and  $S$  to  $X$  along  $M_{L \times \{0\}}$  and  $M_{L_{BD} \times \{0\}}$  respectively. Thus  $E$  is a 4-manifold with boundary  $M_{BD(L)}$ .

We claim that  $E$  is actually an  $(n+1)$ -solution for  $BD(L)$ . We start first by showing  $E$  is an  $(n)$ -solution. Let  $\overline{E} = X \cup W$ . Consider the following long exact sequence (with  $\mathbb{Z}$ -coefficients in reduced homology) obtained by Mayer Vietoris.

$$\begin{aligned} \dots &\xrightarrow{\partial_*} H_2(M_L) \xrightarrow{I_1} H_2(X) \oplus H_2(W) \xrightarrow{I_2} H_2(\overline{E}) \xrightarrow{\partial_*} \\ &H_1(M_L) \xrightarrow{I_1} H_1(X) \oplus H_1(W) \xrightarrow{I_2} H_1(\overline{E}) \xrightarrow{\partial_*} 0. \end{aligned}$$

We have that inclusion induces an isomorphism  $H_1(M_L) \cong H_1(W)$ . This together with the fact that  $I_2$  on  $H_1$  is surjective and  $H_1(M_L) \rightarrow H_1(X)$  is the zero map, gives that  $H_1(\overline{E}) \cong H_1(X)$ . From Lemma 4.12, the inclusion maps induce an isomorphism  $H_2(X) \cong H_2(M_{LBD}) \oplus H_2(M_L)$ . Thus, by the first isomorphism theorem, we obtain the following.

$$\begin{aligned} H_2(\overline{E}) &\cong \frac{H_2(X) \oplus H_2(W)}{\ker(I_2 : H_2(X) \oplus H_2(W) \rightarrow H_2(\overline{E}))} \\ &\cong \frac{H_2(X) \oplus H_2(W)}{\text{image}(I_1 : H_2(M_L) \rightarrow H_2(X) \oplus H_2(W))} \\ &\cong H_2(M_{LBD}) \oplus H_2(W). \end{aligned}$$

Notice that  $E = \overline{E} \cup S$ . Consider the following long exact sequence on homology given by Mayer Vietoris.

$$\begin{aligned} \dots &\xrightarrow{\partial_*} H_2(M_{LBD}) \xrightarrow{\rho_1} H_2(\overline{E}) \oplus H_2(S) \xrightarrow{\rho_2} H_2(E) \xrightarrow{\partial_*} \\ &H_1(M_{LBD}) \xrightarrow{\rho_1} H_1(\overline{E}) \oplus H_1(S) \xrightarrow{\rho_2} H_1(E) \xrightarrow{\partial_*} 0. \end{aligned}$$

Using the facts,  $H_1(M_{LBD}) \cong H_1(X)$  induced by inclusion (Lemma 4.12),  $H_2(S) = 0$ , and  $H_1(X) \cong H_1(\overline{E})$ , we can again use the first isomorphism theorem to attain the following.

$$\begin{aligned}
H_2(E) &\cong \frac{H_2(\overline{E})}{\ker(\rho_2)} \\
&\cong \frac{H_2(M_{LBD}) \oplus H_2(W)}{\text{image}(\rho_1 : H_2(M_{LBD}) \rightarrow H_2(\overline{E}) \oplus H_2(S))} \\
&\cong H_2(W).
\end{aligned}$$

This shows that the second condition of  $(n)$ -solvability is satisfied for the 4-manifold  $E$ . Using the same arguments found in Proposition 2.8, the third condition is also met.

To check the first condition, consider again the long exact sequence (4.3). The first isomorphism theorem tells us the following.

$$\frac{H_1(M_{LBD})}{\ker(\rho_1 : H_1(M_{LBD}) \rightarrow H_1(\overline{E}) \oplus H_1(S))} \cong \text{image}(\rho_1)$$

Since the  $\ker(\rho_1) = \{0\}$ , we have that  $\text{image}(\rho_1) \cong H_1(M_{LBD})$ . Now,  $S$  is an  $(n)$ -solution for  $M_{LBD}$ , thus  $H_1(M_{LBD}) \cong H_1(S)$  induced by inclusion. Using the first isomorphism a final time gives that  $H_1(E) \cong H_1(\overline{E})$ .

By Lemma 4.12 and the above results, the first condition to being  $(n)$ -solvable is met and  $E$  is an  $(n)$ -solution for  $BD(L)$ .

We claim further that  $E$  is actually an  $(n+1)$ -solution. Showing that  $\pi_1(W) \subset \pi_1(E)^{(1)}$  (or more precisely,  $i_*(\pi_1(W)) \subset \pi_1(E)^{(1)}$ ) is enough to imply that  $\pi_1(W)^{(n)} \subset \pi_1(E)^{(n+1)}$ .

Consider the following commutative diagram of maps where  $i_*$  is induced by in-

clusion and both  $p_{1*}$  and  $p_{2*}$  are the canonical quotient maps.

$$\begin{array}{ccc}
 \pi_1(W) & \xrightarrow{i_*} & \pi_1(E) \\
 p_{1*} \downarrow & \searrow h & \downarrow p_{2*} \\
 H_1(W) = \frac{\pi_1(W)}{\pi_1(W)^{(1)}} & \xrightarrow{i_*} & \frac{\pi_1(E)}{\pi_1(E)^{(1)}} = H_1(E)
 \end{array}$$

Showing that  $h \equiv 0$  is equivalent to showing that  $\pi_1(W) \subset \pi_1(E)^{(1)}$ . Examining this further shows that  $h \equiv 0$  if and only if  $i_* : H_1(W) \rightarrow H_1(E)$  is the zero map, since our diagram commutes. Consider the following commutative diagram.

$$\begin{array}{ccc}
 H_1(M_L) & \xrightarrow{\cong} & H_1(W) \\
 & \searrow p & \downarrow i_* \\
 & & H_1(E)
 \end{array}$$

To show that  $i_* \equiv 0$  is equivalent to showing that the map  $p : H_1(M_L) \rightarrow H_1(E)$  is the zero map. Consider  $[\mu_i] \in H_1(M_L)$  where  $\mu_i$  generate  $H_1(M_L)$ . Under the map  $p$ ,  $[\mu_i] = [\eta_i] \in H_1(M_{LBD}) \subset H_1(E)$  ( $\eta_i \in S^3 - L_{BD} \subset M_{LBD}$ ). But recall that  $[\eta_i]$  lie in a commutator subgroup and thus  $[\eta_i] = 0$  in homology, and  $p$  is the zero map. Thus  $E$  is an  $(n + 1)$ -solution and the desired result is achieved.

The case when  $L$  is  $(n.5)$ -solvable is similar. □

# Chapter 5

## Applications to $\{\mathcal{F}_n^m\}$

In studying the  $(n)$ -solvable filtration, we often look at successive quotients of the filtration. Recently, progress has been made towards understanding the structure of its quotients (see [Cha10], [CH08], [CHL09], [Har08]). We will mostly focus on the filtration of  $\mathcal{C}^m$  when  $m \geq 2$ .

Most of the previous work studies the filtration of boundary links. Harvey first showed that  $\mathcal{F}_n^m/\mathcal{F}_{n+1}^m$  is a nontrivial group that contains an infinitely generated subgroup [Har08]. She showed that this subgroup is generated by boundary links (links with components that bound disjoint Seifert surfaces). Boundary links have vanishing  $\bar{\mu}$ -invariants at all lengths. Cochran and Harvey improved this result by showing that  $\mathcal{F}_n^m/\mathcal{F}_{n.5}^m$  contains an infinitely generated subgroup [CH08]. Again, this subgroup consists entirely of boundary links.

Using the relationship between Milnor's  $\bar{\mu}$ -invariants and  $(n)$ -solvability, given in Theorem 3.7, we are able to establish new results that are disjoint from previous work. In addition, we will also investigate the commutivity of these quotient groups.

Until now, nothing has been known about the “other half” of the  $(n)$ -solvable filtration,  $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ .

**Theorem 5.1.**  $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$  contains an infinite cyclic subgroup for  $m \geq 3 * 2^{n+1}$ .

*Proof.* Let  $BR$  = the Borromean Rings, as seen in Example 3.2. We have shown that  $BR$  is a nontrivial link in  $\mathcal{F}_{-0.5}^3/\mathcal{F}_0^3$ . When we take the Bing double of  $BR$ ,  $BD(BR)$  (see Figure 5.1(a)), this new link is in  $\mathcal{F}_{0.5}^6$  by Proposition 4.10. However, the first nonvanishing  $\bar{\mu}$ -invariant is  $\bar{\mu}_{BD(BR)}(I) = \pm 1$  for a certain  $I$  with  $|I| = 6$  (see Chapter 8 in [Coc90] for more details), so  $BD(BR)$  is not (1)-solvable by Theorem 3.7. Then  $BD(BR)$  is nontrivial in  $\mathcal{F}_{0.5}^6/\mathcal{F}_1^6$  since it has a nonvanishing  $\bar{\mu}$ -invariant.

We can perform the Bing doubling operation on this new link to form  $BD(BD(BR))$ , or more simply,  $BD_2(BR)$  (see Figure 5.1(b)). Using Proposition 4.11, we see that  $BD_2(BR)$  is nontrivial in  $\mathcal{F}_{1.5}^{12}$ . Looking at its  $\bar{\mu}$ -invariants, we will have that  $\bar{\mu}_{BD_2(BR)}(I) = \pm 1$  for a certain  $I$  of length 12 and our link cannot be (2)-solvable by Theorem 3.7. Therefore  $BD_2(BR)$  is nontrivial in  $\mathcal{F}_{1.5}^{12}/\mathcal{F}_2^{12}$ . We can continue this process to have  $BD_{n+1}(BR)$  nontrivial in  $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$  for  $m \geq 3 * 2^{n+1}$ .

We claim that  $BD_{n+1}(BR)$  will have infinite order in  $\mathcal{F}_{n.5}^m/\mathcal{F}_{n+1}^m$ . Recall that Orr showed the first nonvanishing  $\bar{\mu}$ -invariant is additive, see Theorem 3.6. We will look at an arbitrary string link  $L$  with the following properties instead of the specific link  $BD_{n+1}(BR)$  for the moment. Suppose that  $\bar{\mu}_L(I) = 0$  and that  $\bar{\mu}_L(J) \neq 0$  for  $|J| = |I| + 1$ . Then

$$\bar{\mu}_{\widehat{LL}}(J) = \bar{\mu}_{\widehat{L}}(J) + \bar{\mu}_{\widehat{L}}(J) = 2\bar{\mu}_{\widehat{L}}(J).$$

If we were to take the closure of the stack of  $n$  copies of  $L$ , denoted  $\widehat{nL}$ , we would



obtain

$$\widehat{\bar{\mu}}_{nL}(J) = n\bar{\mu}_L.$$

Thus  $L$  generates an infinite cyclic subgroup  $\mathbb{Z}$ . In our case, since  $BD_{n+1}(BR)$  has a nonzero  $\bar{\mu}$ -invariant, we can use the same reasoning to show that it generates an infinite cyclic subgroup.  $\square$

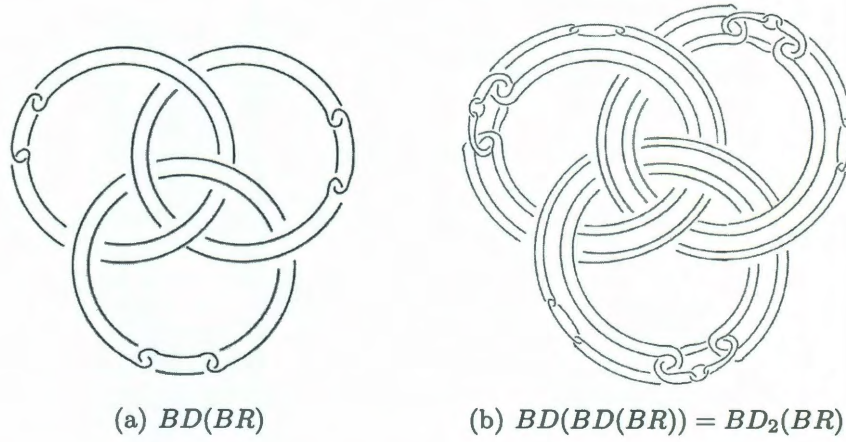


Figure 5.1: Examples of iterated Bing doubles of the Borromean Rings

The example exhibited in the above proof came from iterated Bing doubles of a link with nonvanishing  $\bar{\mu}$ -invariants. These iterated Bing doubles will always have a nonzero  $\bar{\mu}$ -invariant [Coc90]. Thus the above example is also not concordant to a boundary link.

Since the knot concordance group  $\mathcal{C}$  is abelian, all successive quotients of the  $(n)$ -solvable filtration are abelian. It is known, however, that  $\mathcal{C}^m$  is a nonabelian group for  $m \geq 2$  [LD88]. We briefly recall some facts known about certain quotient groups of  $\{\mathcal{F}^m\}$ .

The quotient  $\mathcal{F}_{-0.5}^m / \mathcal{F}_0^m$  is abelian for all  $m \geq 2$ . This arises from the fact there

exists a monomorphism

$$\mathcal{F}_{-0.5}^m / \mathcal{F}_0^m \hookrightarrow \Omega_3^{\text{spin}}(S^1 \times \cdots \times S^1)$$

where  $\Omega_3^{\text{spin}}$  is the 3-dimensional spin bordism group. This group is known to be abelian, giving us that  $\mathcal{F}_{-0.5}^m / \mathcal{F}_0^m$  is abelian.

We also know that the quotient  $\mathcal{C}^m / \mathcal{F}_0^m$  is a nonabelian group for  $m \geq 2$ . We will give an example for  $m \geq 3$ .

**Example 5.2.** Consider the pure braids in Figure 5.2(a) and 5.2(b). A *pure braid* is a string link with all the strings strictly descending. We build the commutator  $ABA^{-1}B^{-1}$  seen in Figure 5.2(c) and check whether or not it is (0)-solvable. Now  $\widehat{ABA^{-1}B^{-1}}$  is isotopic to the Borromean Rings that were shown in Example 3.2 not to be (0)-solvable.

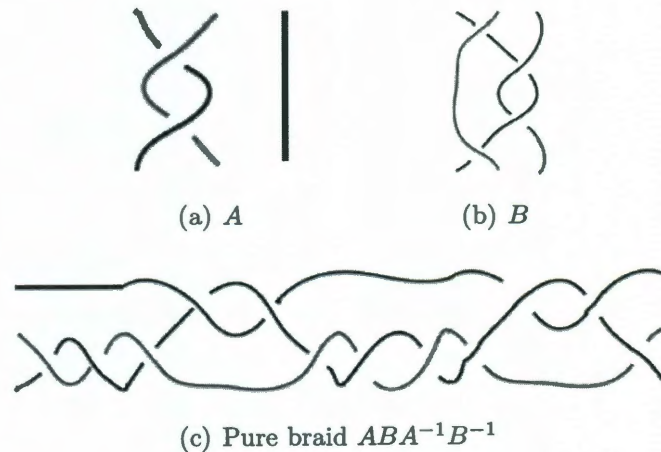


Figure 5.2: Example of pure braids with their commutator not (0)-solvable

We continue on with our investigation of quotients of  $\{\mathcal{F}_n^m\}$ . Again using Theorem 3.7, we will show that  $\mathcal{F}_{-0.5}^m / \mathcal{F}_1^m$  is nonabelian.

**Theorem 5.3.**  $\mathcal{F}_{-0.5}^m/\mathcal{F}_1^m$  is nonabelian for  $m \geq 3$ .

In order to prove this theorem, we need to demonstrate that there exists two string links with pairwise linking numbers equal to zero such that when we construct the commutator we get a string link that is not (1)-solvable.

*Proof.* The Borromean Rings,  $BR$ , can be written as a pure braid, specifically,  $BR = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$  (see Figure 5.3(a)). Consider the pure braid  $\sigma_1 BR\sigma_1^{-1}$ , the Borromean Rings conjugated by  $\sigma_1$  (see Figure 5.3(b)). We look at the commutator  $L = (BR)(\sigma_1 BR\sigma_1^{-1})(BR)^{-1}(\sigma_1 BR\sigma_1^{-1})^{-1}$ . Notice that  $L$  is also a pure braid.

For braids, the canonical meridians,  $m_i$ , will freely generate the fundamental group and any other meridian of  $L_i$  (the  $i^{\text{th}}$  string of  $L$ ) in  $\pi_1$  will be a conjugate of  $m_i$ . This allows us to write  $l_i$  of  $\hat{L}$  as a product of the  $m_i$ 's using an algorithmic procedure.



(a)  $BR$  as a pure braid



(b)  $BR = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$

Figure 5.3: Borromean Rings as a pure braid and a conjugate of them

Christopher Davis wrote a computer program that does exactly that [Dav]. In addition to writing the  $l_i$ 's as products of meridians, this program also computes the Magnus expansion of  $l_i$ . Using this program, we found that there is a  $\bar{\mu}$ -invariant of length six that does not vanish. More specifically,  $\bar{\mu}_L(313323) = -1$ . By Theorem 3.7,  $L$  is not (1)-solvable. Therefore  $\mathcal{F}_{-0.5}^m/\mathcal{F}_1^m$  is nonabelian.  $\square$

# Chapter 6

## Grope Filtration

### 6.1 Grope Filtration

In addition to defining the  $(n)$ -solvable filtration, Cochran, Orr, and Teichner [COT03] also defined the Grope filtration,  $\{\mathcal{G}_n^m\}$  of the (string) link concordance group,

$$\{0\} \subset \cdots \subset \mathcal{G}_{n+1}^m \subset \mathcal{G}_{n.5}^m \subset \mathcal{G}_n^m \subset \cdots \subset \mathcal{G}_{0.5}^m \subset \mathcal{G}_0^m \subset \mathcal{C}^m.$$

The Grope filtration is more geometric than the  $(n)$ -solvable filtration. Gropes can be thought of as geometric approximations of slicing disks.

**Definition 6.1.** A **grope** is a special pair (2-complex, base circle) which has a height  $n \in \frac{1}{2}\mathbb{N}$  assigned to it. A grope of height 1 is precisely a compact, oriented surface  $\Sigma$  with a single boundary component, which is the base circle (see Figure 6.1).

A grope of height  $n + 1$  can be defined recursively by the following construction. Let  $\{\alpha_i, \beta_i : i = 1, \dots, 2(g - 1)\}$  be a symplectic basis of curves for  $H_1(\Sigma)$ , where  $\Sigma$  is

the first stage grope. Then a grope of height  $n + 1$  is formed by attaching gropes of height  $n$  to each  $\alpha_i$  and  $\beta_i$  along the base circles (see Figure 6.1). A grope of height 1.5 is a surface with surfaces attached to ‘half’ of the basis curves (say the  $\alpha_i$ ). A grope of height  $n + 1.5$  is obtained by gluing gropes of height  $n$  to the  $\alpha_i$  and gropes of height  $n + 1$  to the  $\beta_i$ .

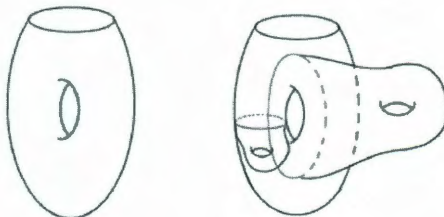


Figure 6.1: A height 1 and height 2 grope

Given a 4-manifold,  $W$ , with boundary  $M$  and a framed circle  $\gamma \subset M$ , we say that  $\gamma$  bounds a **Grope** in  $W$  if  $\gamma$  extends to a smooth embedding of a grope with its untwisting framing (parallel push offs of Gropes can be taken in  $W$ ).

We denoted  $\mathcal{G}_n^m$  to be the subset of  $\mathcal{C}^m$  defined by the following. A link  $L$  is in  $\mathcal{G}_n^m$  if the components of  $\hat{L}$  bound disjoint Gropes of height  $n$  in  $D^4$ . It can be shown that these subsets are actually normal subgroups of  $\mathcal{C}^m$ . Harvey showed [CH08] that this filtration is nontrivial by looking at the filtration of boundary string links.

A useful property of Gropes is demonstrated in the following lemma.

**Lemma 6.2.** *If a curve  $\ell$  bounds a (map of a) grope of height  $n$  in a space  $X$ , then  $[\ell] \in \pi_1(X)^{(n)}$ .*

*Proof.* Suppose that  $n = 1$ . Let  $\Sigma$  be a grope of height 1 that the curve  $\ell$  bounds in some space  $X$ . Then  $\ell$  can be written as a commutator of  $\alpha$  and  $\beta$ , see Figure 6.2. Thus  $\ell = \alpha\beta\alpha^{-1}\beta^{-1}$  and  $[\ell] \in \pi_1(X)^{(1)}$ .

Now suppose that if  $\ell$  bounds a grope of height  $k$  then  $[\ell] \in \pi_1(X)^{(k)}$  for all  $k < n$ . Assume that  $\ell$  bounds a height  $n$  grope. Then  $\ell = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$  where  $\alpha_i$  and  $\beta_i$  form a symplectic basis of  $\Sigma$ , the first stage grope. Since  $\alpha_i$  and  $\beta_i$  bound  $(n-1)$ -stage gropes, by the induction hypothesis,  $\alpha_i, \beta_i \in \pi_1(X)^{(n-1)}$ . Thus  $[\ell] \in \pi_1(X)^{(n)}$ .

□

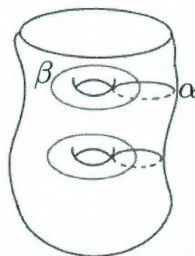


Figure 6.2: Basis curves  $\alpha$  and  $\beta$

There is also a notion of Grope concordance. To define this, the following definition is needed.

**Definition 6.3.** An **annular grope** of height  $n$  is a grope of height  $n$  that has an extra boundary component on its first stage.

The two boundary components of an annular grope are said to cobound an annular grope. Two links,  $L_0$  and  $L_1$ , are *height  $n$  Grope concordant* if their components cobound disjoint height  $n$  annular Gropes,  $G_i$ , in  $S^3 \times [0, 1]$  such that  $G_i \cap (S^3 \times \{j\}) =$  the  $i^{\text{th}}$  component of  $L_j$  where  $j = 0, 1$ .

## 6.2 Relationship between Filtrations

Thus far, two filtrations of the string link concordance group  $\mathcal{C}^m$  have been defined. Recall that the  $(n)$ -solvable filtration is an algebraic approximation while the Grope

filtration is a geometric approximation. It is a natural question to ask whether these two filtrations are related. Before answering this question, we need to look at the relationship between a link bounding disjoint gropes and the link's Milnor's  $\bar{\mu}$ -invariants.

**Definition 6.4.** Let  $L = L_1 \cup L_2 \cup \cdots \cup L_m$  and  $L' = L'_1 \cup L'_2 \cup \cdots \cup L'_m$  be ordered, oriented links in  $S^3$ . We say that  $L$  is  $k$ -cobordant to  $L'$ , where  $k \in \mathbb{Z}^+$ , if there are disjointly embedded compact, connected, oriented surfaces  $V_1, V_2, \dots, V_m$  in  $S^3 \times [0, 1]$  with  $\partial V_i = \partial_0 V_i \cup \partial_1 V_i$  such that for all  $i = 1, \dots, m$ , we have

- i.  $V_i \cap (S^3 \times \{0\}) = \partial_0 V_i = L_i$  and  $V_i \cap (S^3 \times \{1\}) = \partial_1 V_i = L'_i$ ;
- ii. there is a tubular neighborhood  $V_i \times D^2$  of  $V_i$  in  $S^3 \times [0, 1]$  which extends the “longitudinal” ones of  $\partial V_i = L_i \cup L'_i$  in  $S^3 \times \{0\}$  and  $S^3 \times \{1\}$  resp such that the image of the homomorphism

$$\pi_1(V_i) \rightarrow \pi_1(V_i \times \partial D^2) \rightarrow \pi_1(S^3 \times [0, 1] - V) = G$$

lies in the  $k$ th term of the lower central series of  $G$ .

A link that is  $k$ -cobordant to a slice link is called **null  $k$ -cobordant**.

Links that bound disjoint Gropes of height  $n$  will be  $k$ -cobordant to a slice link for certain  $k$  dependent on  $n$ .

**Proposition 6.5.** *If  $L \in \mathcal{G}_n^m$ , then it is  $2^{n-1}$ -cobordant to a slice link.*

*Proof.* Suppose  $L \in \mathcal{G}_{n+2}^m$ . Then the components of  $L$ , say  $\ell_i$ , bound disjoint Gropes of height  $n$  in  $D^4 \cong S^3 \times [0, 1]$ . Moreover, the  $\ell_i$ s extend to smooth embeddings of

gropes with their untwisting framing. Also,  $L$  is Grope concordant to a slice link  $L'$ . Let  $V_i$  be the first stage Grope bounded by  $\ell_i$  and  $\ell'_i$  (ie. the annular Grope in the concordance). Let  $V = \coprod_{i=1}^m V_i$ .

Now consider the homomorphism

$$\pi_1(V_i) \rightarrow \pi_1(V_i \times \partial D^2) \rightarrow \pi_1(S^3 \times [0, 1] - V) = G$$

that is induced by pushing  $V_i$  off itself in the normal direction. Let  $\{\alpha_i, \beta_i\}$  be a symplectic basis for  $V_i$  (see Figure 6.3). The parallel push-offs of Gropes can be taken in  $S^3 \times [0, 1]$  and thus are now in  $S^3 \times [0, 1] - V$ . We seek to find the images of the basis elements under the above homomorphism. By the construction of the Gropes, each of the  $\alpha_i$ s and  $\beta_i$ s bound Gropes of height  $n - 1$  in the exterior of  $V$ . Thus

$$[\alpha_i], [\beta_i] \in G^{(n-1)} \subset G_{2n-1}$$

by Lemma 6.2 and this concludes the proof. □

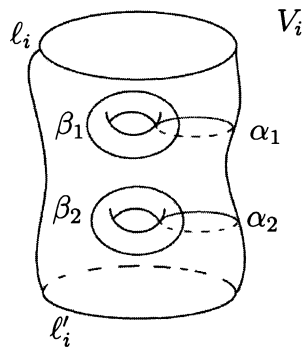


Figure 6.3: The first stage grope,  $V_i$  with symplectic basis  $\{\alpha_i, \beta_i\}_{i=1,2}$ .

The following corollary of Lin [Lin91] can be applied to give a nice result.



**Corollary 6.6** (Lin). *If  $L$  and  $L'$  are  $k$ -cobordant, then Milnor's  $\bar{\mu}$ -invariants of  $L$  and  $L'$  with lengths less than or equal to  $2k$  are the same. In particular, if  $L$  is null  $k$ -cobordant, then  $\bar{\mu}_L(I) = 0$  for  $|I| \leq 2k$ .*

**Corollary 6.7.** *A link  $L$  with components that bound disjoint Gropes of height  $n$  has  $\bar{\mu}_L(I) = 0$  for  $|I| \leq 2^n$ .*

*Proof.* The proof of this is immediate from the previous two results.  $\square$

Cochran, Orr and Teichner [COT03] showed that these two filtrations are related. More specifically, that we have inclusion in one direction.

**Theorem 6.8** (Cochran-Orr-Teichner). *If a link  $L$  bounds a grope of height  $n + 2$  in  $D^4$ , then  $L$  is  $(n)$ -solvable, i.e.  $\mathcal{G}_{n+2}^m \subseteq \mathcal{F}_n^m$  for all  $m$  and  $n$ .*

The natural question is whether or not the inclusion goes in the other direction. In other words, if a link is  $(n)$ -solvable, do the components bound disjoint Gropes of height  $n + 2$ ? Recall from Theorem 3.7 that an  $(n)$ -solvable link has vanishing  $\bar{\mu}$ -invariants for lengths less than or equal to  $2^{n+2} - 1$ , while above we see that a link in  $\mathcal{G}_{n+2}^m$  has vanishing  $\bar{\mu}$ -invariants for lengths less than or equal to  $2^{n+2}$ . This difference of one gives motivation to try to find a nontrivial element in  $\mathcal{F}_n^m / \mathcal{G}_{n+2}^m$ .

**Corollary 6.9.**  *$\mathcal{F}_n^m / \mathcal{G}_{n+2}^m$  is nontrivial for  $m \geq 2^{n+2}$ . Moreover,  $\mathbb{Z} \subset \mathcal{F}_n^m / \mathcal{G}_{n+2}^m$ .*

*Proof.* Let  $L$  be the Hopf link. By Proposition 4.7,  $BD(L) \in \mathcal{F}_0$ , where  $BD(L)$  is the Bing double  $L$  (see Figure 4.5(d)). The invariant  $\bar{\mu}_L(12) = \pm 1$  depending on orientation, as it is just the linking number between the two components. Again, by work Cochran given in Chapter 8 of [Coc90], (Theorem ??),  $\bar{\mu}_{BD(L)}(I) = \pm 1$  for some

$I$  of length 4. Using iterated Bing doubling we achieve  $BD_{n+1}(L)$  is nontrivial in  $\mathcal{F}_n$  by Proposition 4.11, and  $\bar{\mu}_{BD_{n+1}(L)}(I) = \pm 1$  for some  $I$  of length  $2^{n+2}$ .  $BD_{n+1}(L)$  is  $(n)$ -solvable, but since some  $\bar{\mu}_{BD_{n+1}(L)}$  does not vanish for a length of  $2^{n+2}$  it cannot bound a Grope of height  $n + 2$ .

To show that there is an infinite cyclic subgroup contained within this quotient, we look to the proof in Theorem 5.1 for a completely analogous argument.  $\square$

This tells us that the Grope filtration and  $(n)$ -solvable filtration of  $\mathcal{C}^m$  are not the same (for  $m \geq 2^{n+2}$ ).

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