Math 101: Calculus 1
Instructor: Taylor Coon
Midterm Exam 1
October 4, 2010

Please print name: **Answer key**

When you are finished with the exam, please sign the honor code below.
On my honor, I have neither given nor received any aid on this exam:

No books, notes, or calculators may be used on this exam. There are 6 problems and one bonus problem. You must show all of your work to receive full credit. Answers without any supporting work will receive little credit.

The bonus problem may be submitted with your exam for up to 10 points of extra credit. If you do not get to try the bonus problem during the test period, you may take the last page of the exam with you and turn in the bonus problem at the beginning of class on Wednesday, October 6 for up to 5 extra credit points. The bonus problem is pledged, so you may not consult your book, notes, classmates, internet, or any other sources to complete it.

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(#1: 10 pts) Find the limit:

a. \[
\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 + x - 6}
\]
\[
= \lim_{x \to 2} \frac{(x+5)(x-2)}{(x+3)(x-2)}
\]
\[
= \lim_{x \to 2} \frac{x+5}{x+3}
\]
\[
= \frac{2+5}{2+3} = \frac{7}{5}
\]

b. \[
\lim_{x \to \infty} \sqrt{x^2 + 5} - x
\]
\[
= \lim_{x \to \infty} \frac{\sqrt{x^2 + 5} - x \cdot \sqrt{x^2 + 5} + x}{\sqrt{x^2 + 5} + x}
\]
\[
= \lim_{x \to \infty} \frac{(x^2 + 5) - x^2}{\sqrt{x^2 + 5} + x}
\]
\[
= \lim_{x \to \infty} \frac{5}{\sqrt{x^2 + 5} + x}
\]
\[
= \frac{5}{\infty} = 0
\]
(20 pts) a. Graph the function $f(x) = 3(x-1)^4 + 4$. (Hint: use stretching and shifting techniques from class). Make sure to label your axes.

b. Identify a point on your graph where $f' = 0$.

$\text{when } x = 1, \text{ horizontal tangent means } f'(1) = 0$

c. Identify an interval on which $f'$ is positive.

$(1, \infty)$

d. Using this information, sketch a graph of $f'(x)$.

$\text{f}'(1)=0$  
$\text{f}' > 0 \text{ when } x > 1$  
$\text{f}' < 0 \text{ when } x < 1$  

Tangent lines get steep quickly as $x \to \infty$.

e. Is $f'(x)$ one-to-one? How do you know?

Yes. It's graph passes the horizontal line test.
f. For \( f(x) = 3(x - 1)^4 + 4 \), compute \( f'(x) \) and use this equation to find \( f'^{-1}(x) \).

\[
f'(x) = 4 \cdot 3(x - 1)^3
\]

\[
= 12(x - 1)^3
\]

\[
y = 12(x - 1)^3
\]

\[
\frac{y}{12} = (x - 1)^3
\]

\[
\frac{3\sqrt{y}}{12} = x - 1
\]

\[
\Rightarrow f'^{-1}(x) = \frac{3\sqrt{x}}{12} + 1
\]

\[
g(x) = \begin{cases} 
\sin(x) & x \geq 0 \\
1 - \cos(2x) & x < 0 
\end{cases}
\]

Is \( g(x) \) continuous at 0? Is \( g(x) \) differentiable at 0? Why or why not?

\begin{itemize}
  \item \( g(x) \) is continuous at 0 if \( \lim_{x \to 0} g(x) = g(0) \).

\[
\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \sin(x) = \sin(0) = 0
\]

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1 - \cos(2x) = 1 - \cos(0) = 0
\]

Since these limits agree, \( g \) is continuous at 0.

\item \( g(x) \) is differentiable if \( g'(0) \) exists.

\[
\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} (\sin(x))' = \lim_{x \to 0^+} \cos(x) = 1
\]

\[
\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} (1 - \cos(2x))' = \lim_{x \to 0^-} 2\sin(2x) = 2\sin(0) = 0
\]

Since \( 1 \neq 0 \), \( g'(0) \) does not exist, so \( g \) is NOT differentiable at 0.
(#4: 10 pts) Find the derivative of:

a. \( p(x) = \tan^2\left(\frac{1}{\sqrt{x+1}}\right) \) chain rule!

\[
p'(x) = 2\tan\left(\frac{1}{\sqrt{x+1}}\right) \cdot \sec^2\left(\frac{1}{\sqrt{x+1}}\right) \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{x+1}}\right) \\
= 2\tan\left(\frac{1}{\sqrt{x+1}}\right) \sec^2\left(\frac{1}{\sqrt{x+1}}\right) \left(-\frac{1}{2}\right) (x+1)^{-3/2} \\
= -\tan\left(\frac{1}{\sqrt{x+1}}\right) \sec^2\left(\frac{1}{\sqrt{x+1}}\right) (x+1)^{3/2}
\]

b. \( q(x) = \frac{x^3}{x^4-4} \) quotient rule!

\[
q'(x) = \frac{\left(\frac{d}{dx}(x^3)(x^2+4) - (x^3)(\frac{d}{dx}(x^2-4))\right)}{(x^2-4)^2} \\
= \frac{(3x^2+x^3 \ln 3)(x^2+4) - (x^3)(2x)}{(x^2-4)^2} \\
= \frac{(3x^2+x^3 \ln 3)(x^2+4) - 2x^2 3^x}{(x^2-4)^2}
\]

Note: Some of you did this problem using logarithmic differentiation. That works, too; as long as you did it correctly, you get full credit.
(#5: 10 pts) The function \( f(x) \) is continuous on the interval \([0, 1]\). \( f(0) = 1 \) and \( f(1) = \frac{1}{2} \). Using the Intermediate Value Theorem, show that there must be some number \( x \) in the interval \((0, 1)\) for which \( f(x) = x \). (Hint: Drawing a picture might help.)

The IVT says that for \( f(x) \) continuous on \([0, 1]\) with \( f(0) = 1 \) and \( f(1) = \frac{1}{2} \), for any \( N \) in the interval \([\frac{1}{2}, 1]\), there is some \( c \) in \([0, 1]\) so that \( f(c) = N \).

This isn't quite enough to give us what we want. Instead, let's define the function \( g(x) = f(x) - x \). Then, \( g(x) \) is also continuous on \([0, 1]\) (since both \( f(x) \) and \(-x\) are continuous on \([0, 1]\)).

\[
g(0) = f(0) - 0 = 1 \quad \text{and} \quad g(1) = f(1) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}.
\]

So, then the IVT says that for \( g(x) \) continuous on \([0, 1]\) with \( g(0) = 1 \) and \( g(1) = -\frac{1}{2} \), for any \( N \) on \([-\frac{1}{2}, 1]\), there is some \( c \) in \([0, 1]\) so that \( g(c) = N \).

In particular, since \( g(x) \) is in the interval \([-\frac{1}{2}, 1]\), there is some \( c \) in \([0, 1]\) so that \( g(c) = 0 \).

If \( g(c) = 0 \) and \( g(c) = f(c) - c \), then \( f(c) - c = 0 \), so \( f(c) = c \), which is what we wanted to show.
(6: 10 pts) Find the line tangent to the curve 

\[ x^2 - y^3 = \frac{1}{2}xy^2 \]

at the point \((-2, 2)\).

Implicit Differentiation:

\[
2x - 3y^2 \frac{dy}{dx} = \frac{1}{2} \left( y^2 + 2xy \frac{dy}{dx} \right) = \frac{1}{2} y^2 + xy \frac{dy}{dx}
\]

\[
2x - \frac{1}{2} y^2 = xy \frac{dy}{dx} + 3y^2 \frac{dy}{dx}
\]

\[
2x - \frac{1}{2} y^2 = \frac{dy}{dx} \left( xy + 3y^2 \right)
\]

So

\[
\frac{dy}{dx} = \frac{2x - \frac{1}{2} y^2}{xy + 3y^2}
\]

@ \((-2, 2), \quad \frac{dy}{dx} = \frac{2(-2) - \frac{1}{2}(2)^2}{(-2)(2) + 3(2)^2}
\]

\[
= \frac{-4 - \frac{2}{2}}{-4 + 12} = \frac{-6}{8} = -\frac{3}{4}
\]

So slope of this line is \(-\frac{3}{4}\) and the line goes through \((-2, 2)\)

\[
\Rightarrow \text{Tangent line is } y - 2 = \frac{-3}{4} (x + 2)
\]