This take-home exam is intended to be taken in 3 hours. No books, notes, calculators, or any other form of aid is permitted. There are 100 possible points. You must show all of your work, justifying when necessary. If you need more space, attach any additional pages. Partial credit will be given. If you have any questions during the exam, you can reach me at (610) 442 - 4559.

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a. In a second-order linear equation, what is a *forcing term*?

A second-order linear equation is one of the form \( y'' + p(t)y' + q(t)y = f(t) \). The forcing term is the function \( f(t) \).

b. What does it mean for two linear systems to be *equivalent*? Do equivalent systems have to have the same number of equations? Do they have to have the same number of variables?

Two systems are equivalent if they have the same set of solutions. They do not need to have the same number of equations, but they must have the same number of variables.

c. What are the criteria for a set of vectors to be a *basis* for a space \( V \)?

A set of vectors form a basis for \( V \) if they span all of \( V \) and are linearly independent.
(#2: 10 points) Find any particular solution for the differential equation

\[ x^3 dx + (2xy\sqrt{xy})dy = 0 \]

This is like an example we did in class in section 2.6.

\[ P(x, y) = x^3 \] and \[ Q(x, y) = 2xy\sqrt{xy} \] are both homogeneous of degree 3. Thus, we can make the substitution \( y = xv \) and separate the equations.

\[
x^3 dx + (2x^2v\sqrt{x^2v})(xdv + vdx) = 0
\]

\[
x^3 dx + 2x^4v\sqrt{v}dv + 2x^3v^2\sqrt{v}dx = 0
\]

\[
(x^3 + 2x^3v^2\sqrt{v})dx + 2x^4v\sqrt{v}dv = 0
\]

factor out an \( x^3 \)

\[
(1 + 2v^2\sqrt{v})dx + 2xv\sqrt{v}dv = 0
\]

Separate the variables

\[
\frac{dx}{x} = -\frac{2v^{\frac{3}{2}}}{(1 + 2v^2)} dv
\]

Integrate to get

\[
\ln|x| = -\frac{2}{5} \ln|1 + 2v^\frac{5}{2}| + C
\]

\[
x = A(1 + 2v^\frac{5}{2})^{\frac{2}{5}}
\]

So a particular solution is when \( A = 1 \). We make the substitution \( v = y/x \).

\[
x = (1 + 2(y^5/x)^{\frac{5}{2}})^{\frac{2}{5}}
\]
The equation \( y(t) = t^2 \cos(3t) \) is a solution to the differential equation
\[ y'' + 9y = 2 \cos(3t) - 12t \sin(3t) \]

Find the general solution to this equation.

Note: There was a typo on the exam. It is fixed above, but it does not affect the answer to this problem.

The general solution is \( t^2 \cos(3t) + c_1 y_1(t) + c_2 y_2(t) \) where \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the associated homogenous equation.

To find \( y_1 \) and \( y_2 \), we solve the equation \( y'' + 9y = 0 \). This equation has characteristic equation \( \lambda^2 + 9 = 0 \) so \( \lambda = \pm 3i \)

\[ z(t) = e^{3i} = \cos(3t) + i \sin(3t) \]
\[ \bar{z}(t) = e^{-3i} = \cos(3t) - i \sin(3t) \]

Then,
\[ y_1(t) = 2 \cos(3t) \]
\[ y_2(t) = 2 \sin(3t) \]

So the general solution is
\[ y(t) = t^2 \cos(3t) + c_1 \cos(3t) + c_2 \sin(3t) \]
(#4: 10 points) Write the characteristic equation and then find the general solution to the following homogeneous linear equations.

a.

\[ y'' + y' - 12y = 0 \]

\[ \lambda^2 + \lambda - 12 = 0 \]

\[ (\lambda + 4)(\lambda - 3) = 0 \]

\[ \lambda_1 = -4, \lambda_2 = 3 \]

\[ y_1(t) = e^{-4t} \]

\[ y_2(t) = e^{3t} \]

\[ y(t) = c_1 e^{-4t} + c_2 e^{3t} \]

b.

\[ y'' - 8y' + 16y = 0 \]

\[ \lambda^2 - 8\lambda + 16 = 0 \]

\[ (\lambda - 4)^2 = 0 \]

So

\[ y_1(t) = e^{4t} \]

\[ y_2(t) = te^{4t} \]

So the general solution is

\[ y(t) = c_1 e^{4t} + c_2 te^{4t} \]
The functions \( y_1(t) = \cos(4t) \) and \( y_2(t) = \sin(4t) \) form a fundamental set of solutions for the differential equation \( y'' + 16y = 0 \). Using this information, find a particular solution to the equation

\[
y'' + 16y = \tan(4t) \sec^2(4t)
\]

Helpful reminder: \((\sec(x))' = \sec(x) \tan(x)\)

We use variation of parameters to find a solution of the form

\[
y(t) = v_1(t) \cos(4t) + v_2(t) \sin(4t)
\]

\[
y' = v_1' \cos(4t) - 4v_1 \sin(4t) + v_2' \sin(4t) + 4v_2 \cos(4t)
\]

Impose the condition that \( v_1' \cos(4t) + v_2' \sin(4t) = 0 \)

\[
y' = -4v_1 \sin(4t) + 4v_2 \cos(4t)
\]

\[
y'' = -4v_1' \sin(4t) - 16v_1 \cos(4t) + 4v_2' \cos(4t) - 16v_2 \sin(4t)
\]

We plug these into the second order equation:

\[
y'' + 16y = -4v_1' \sin(4t) + 4v_2' \cos(4t) = \sec^2(4t) \tan(4t)
\]

Using the second equation, we find that:

\[
v_1' = -v_2 \frac{\sin(4t)}{\cos(4t)}
\]

Substituting into the second equation, we have:

\[
4v_2' \sin^2(4t) + 4v_2' \cos(4t) = \sec^2(4t) \tan(4t)
\]

\[
\Rightarrow v_2' = \frac{1}{4} \sec(4t) \tan(4t)
\]

\[
\Rightarrow v_2 = \frac{1}{16} \sec(4t)
\]

\[
\Rightarrow v_1' = -\frac{1}{4} \sec(4t) \tan^2(4t)
\]

The general solution is \( y(t) = -\frac{1}{4} \cos(4t) \int \sec(4t) \tan^2(4t) + \frac{1}{16} \sec(4t) \sin(4t) \)
(#6: 10 points) Find a basis for the nullspace of the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & -2 \\ 1 & 3 & 4 & 0 \end{pmatrix}$$

We put the matrix in row echelon form. Switch rows one and two

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ -2 & 1 & 0 & -2 \end{pmatrix}$$

Row 2 + twice Row 1

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 7 & 8 & -2 \end{pmatrix}$$

$$x_1 + 3x_2 + 4x_3 = 0$$
$$7x_2 + 8x_3 - 2x_4 = 0$$

We have free variables $$x_3$$ and $$x_4$$

Therefore, the nullspace of $$A$$ is

$$\left\{ \begin{pmatrix} -2a - \frac{2}{7}b \\ -\frac{3}{7}a + \frac{4}{7}b \\ a \\ b \end{pmatrix} \right\}$$

A basis for the nullspace is

$$\left\{ \begin{pmatrix} -\frac{4}{7} \\ -\frac{3}{7} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{6}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right\}$$
Let \( V = \text{span} \left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \)

a. Discuss the space \( V \) using terminology from class.

\( V \) is the set of all linear combinations of the vectors \( \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \).

b. Write \( V \) in set notation.

\[
V = \left\{ a \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}
\]

For \( a \) and \( b \) real numbers.

c. Give a basis for the space \( V \).

Since \( \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) are linearly independent, they form a basis for \( V \).

d. Is \( V = \mathbb{R}^3 \)? Why or why not? If not, find a vector in \( \mathbb{R}^3 \) that is not in \( V \).

No. Two vectors in \( \mathbb{R}^3 \) can’t span all of \( \mathbb{R}^3 \). There are many vectors not in \( V \). One such vector is \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
(\#8: 10 points)
a. Are the vectors \( \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \) linearly independent?

The vectors are linearly dependent. We can tell this by putting the matrix

\[
\begin{pmatrix}
2 & 3 & 2 \\
1 & 2 & 2 \\
-2 & -2 & 0
\end{pmatrix}
\]

in row echelon form and seeing that it has a free variable.

b. Is the vector \( \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \) in \( \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \)?

No. We can put the system in row echelon form and find that it is inconsistent.

\[
\begin{pmatrix}
2 & 3 & 2 & 1 \\
1 & 2 & 2 & 0 \\
-2 & -2 & 0 & 4
\end{pmatrix}
\]

Reduced row echelon form is

\[
\begin{pmatrix}
1 & 2 & 2 & 0 \\
0 & -1 & -2 & 1 \\
0 & 0 & 0 & 6
\end{pmatrix}
\]