# Two-dimensional multiple-valued Dirichlet minimizing functions

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**Abstract**

We give some remarks on two-dimensional multiple-valued Dirichlet minimizing functions, including frequency, classification of branch points and their connections. As an application, we prove that blowing-up functions of a two-dimensional multiple-valued Dirichlet minimizing function are unique. This paper is concluded with a boundary regularity theorem for two-dimensional multiple-valued Dirichlet minimizing functions.

## 1 Introduction

Almgren [AF] used multiple-valued Dirichlet minimizing functions to study the interior regularity of mass-minimizing rectifiable currents. One-dimensional multiple-valued Dirichlet minimizing functions have already been characterized as follows:

**Theorem 1.1** ([AF], §2.14)**.** *Suppose $f \in \mathcal{Y}_2((-1,1),\mathbb{Q})$ (see the next section for the definition of $\mathcal{Y}_2$) is strictly defined and Dirichlet minimizing. Then there exists $J \in \{1,2,\cdots,Q\}$, $k_1,k_2,\cdots,k_J \in \{1,2,\cdots,Q\}$ with $Q = k_1+k_2+\cdots+k_J$, and $f_1,f_2,\cdots,f_J \in \mathbb{A}(1,n)$ such that*
*(1) Whenever $-1<x<1$, and $i,j \in \{1,2,\cdots,J\}$ with $i \neq j$, $f_i(x) \neq f_j(x)$.*
*(2) For each $-1<x<1$, $f(x) = \sum_{i=1}^{J} k_i[[f_i(x)]]$.*

So the next natural question is to study the two-dimensional case. In fact, Almgren [AF] has already proven some results regarding this. He introduced frequency function to study the branching behavior of multiple-valued Dirichlet minimizing functions:

$$N(r) := r\int_{\mathbb{B}^m_r(a)} |Df|^2 \Big/ \int_{\partial\mathbb{B}^m_r(a)} |f|^2. \tag{1}$$

For Dirichlet minimizing functions, Almgren (see Theorem 2.3) showed that $N(r)$ is nondecreasing in $r$ by certain range and domain deformations, called "squashing" and "squeezing". The monotonicity property enables one to prove by dimension reduction that such functions have branch sets of codimension at least 2. In two-dimensional case, they are countable (see Theorem 2.5). This result was improved by S. Chang [CS] a few years later. Chang proved that branch points of two-dimensional Dirichlet minimizers are isolated. Almgren also proved that for two-dimensional domain, the frequency of a $Q-$valued Dirichlet minimizer is either 0 or no less than $1/Q$ (see Theorem 2.4).

The very first motivation of this paper was trying to further understand the frequency of two-dimensional minimizers. Thanks to Theorem 2.4, which says that blowing-up at a branch point of a Dirichlet minimizer yields a Dirichlet minimizer which is homogeneous and the degree is the same as the frequency of

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original minimizer at that branch point. Hence the study of frequency amounts to the study of homogeneous Dirichlet minimizers. Examples like $z^{1/2}$ suggests that the frequency cannot be any arbitrary real number. More precisely, if we consider $\mathbf{Q}_2(\mathbb{R}^2)$, any function like

$$f(z) = z^N, \text{ for some positive real number } N$$

could be a candidate (one thing worth mentioning is that the frequency function at the origin of $z^N$ is exactly $N$). However, not every $N$ gives a 2-valued function because the function has to match up with itself once as it goes around the circle one time. For example, consider the function

$$f : (r, \theta) \to [[(r^{1/4}\cos(\theta/4), r^{1/4}\sin(\theta/4))]] + [[(-r^{1/4}\cos(\theta/4), -r^{1/4}\sin(\theta/4))]].$$

$$f(r, 0) = [[(r^{1/4}, 0)]] + [[(-r^{1/4}, 0)]], f(r, 2\pi) = [[(0, r^{1/4})]] + [[(0, -r^{1/4})]].$$

They do not match. We will see that in this case, only by choosing $N = p/2$ for some positive integer $p$ makes $f$ a 2-valued function. Here is the main theorem regarding frequency:

**Theorem 1.2.** *Suppose $f \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q})$ is strictly defined and Dirichlet minimizing, then the frequency function at every point of $\mathbb{U}_1^2(0)$ is either zero or of the form*

$$p/q \text{ for some positive integers } p, q \text{ with } q \leq Q.$$

*In particular, in case $Q = 2$, the frequency function of a two-dimensional 2-valued Dirichlet minimizing function is either zero or $p/2$ for some positive integer $p$.*

In light of the fact that branch points of two-dimensional minimizers are isolated, we investigate the branch points by looking at the local behavior nearby. Roughly speaking, for a two-dimensional 2-valued minimizer, if we assume $f(0) = 2[[0]]$ is a branch point, then when $r > 0$ is small enough, $f|\partial\mathbb{B}_r^2(0)$ consists of either a "two-level parking garage" or two disjoint curves. For the first case, $f$ can be obtained from "wrapping" a single-valued harmonic function, while for the second case, $f$ is the "sum" of two single-valued harmonic functions. See Theorem 4.1 and 4.2 for precise statement.

Blowing-up technique plays an important role in geometric measure theory. By blowing-up a multiple-valued Dirichlet minimizer at a branch point, we obtain a homogeneous Dirichlet minimizer, which in some sense, reduces the dimension by one and allows the dimension reduction argument (see [AF] §2.14 for more detail). Generally, it is not clear whether blowing-up is independent of the re-scaling factors. However, for two-dimensional domain, we are able to prove that blowing-up functions are unique thanks to the local structure theorems 4.1 and 4.2. This sort of has the same spirit as the result regarding the uniqueness of tangent cones for two-dimensional area-minimizing integral currents (see [WB]).

The last section is devoted to the boundary regularity of two-dimensional multiple-valued Dirichlet minimizers. The interior regularity has already been established by Almgren [AF] (see Theorem 2.13 there), which claims that they are Hölder continuous in the interior of the domain. As for the boundary regularity, up to the author's knowledge, nothing has been proved yet. One of the major obstacles here is the lack of suitable notion of "subtraction" between multiple-valued functions. We are not able to convert the boundary problem to interior problem by reflection argument. However, for the two-dimensional case, using the Courant-Lebesgue lemma and maximum principle, we can prove

**Theorem 1.3.** *Suppose $f \in \mathcal{Y}_2(\mathbb{B}_1^{2,+}(0), \mathbb{Q})$ is strictly defined and Dirichlet minimizing. If $f|\mathbb{I}_1$ is continuous, then $f$ extends continuously near the origin.*

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## 2 Preliminaries

In our terminology and notations we shall follow [AF]. Throughout the paper $m, n$ and $Q$ will be positive integers. We denote by $\mathbb{U}_r^m(x)$ and $\mathbb{B}_r^m(x)$ the open and closed balls in $\mathbb{R}^m$ with center $x$ and radius $r$.
**The metric space $\mathbf{Q}_Q(\mathbb{R}^n)$.** The space of unordered $Q$ tuples of points in $\mathbb{R}^n$ is

$$\mathbf{Q}_Q(\mathbb{R}^n) = \left\{ \sum_{i=1}^{Q} [[p_i]] : p_1, \dots, p_Q \in \mathbb{R}^n \right\} \text{(often abbreviated as } \mathbb{Q} \text{ or } \mathbb{Q}_Q\text{)}$$

This space is equipped with the metric $\mathcal{G}$:

$$\mathcal{G}\left( \sum_{i=1}^{Q} [[p_i]], \sum_{i=1}^{Q} [[q_i]] \right) = \min_{\sigma} \left( \sum_{i=1}^{Q} |p_i - q_{\sigma(i)}|^2 \right)^{1/2}$$

where $\sigma$ runs through all the permutations of $\{1, \dots, Q\}$. A multiple-valued function (or simply $Q-$valued function) is a $\mathbb{Q}$-valued function.

There are a bi-Lipschitz homeomorphism $\xi : \mathbb{Q} \to \mathbb{Q}^* \subset \mathbb{R}^{PQ}$, where $P$ is a positive integer, and a Lipschitz retraction $\rho : \mathbb{R}^{PQ} \to \mathbb{Q}^*$ with $\rho|_{\mathbb{Q}^*} = \mathrm{Id}_{\mathbb{Q}^*}$ (see Theorems 1.2(3) and 1.3(1) in [AF]).

**Affine approximations.** The set of all affine maps $\mathbb{R}^m \to \mathbb{R}^n$ is denoted by $\mathbb{A}(m, n)$. If $v = (v_1, \dots, v_n) \in \mathbb{A}(m, n)$, we put

$$|v| = \left( \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \frac{\partial v_i}{\partial x_j} \right)^2 \right)^{1/2} \in \mathbb{R}.$$

A function $v : \mathbb{R}^m \to \mathbb{Q}$ is called affine if there are $v_1, \dots, v_Q \in \mathbb{A}(m, n)$ such that $v = \sum_{i=1}^{Q} [[v_i]]$. Then we set $|v| = \left( \sum_{i=1}^{Q} |v_i|^2 \right)^{1/2}$.

If $\Omega \subset \mathbb{R}^m$ is open and $a \in \Omega$, a function $u : \Omega \to \mathbb{Q}$ is said to be approximately affinely approximatable at $a$ if there is an affine function $v : \mathbb{R}^m \to \mathbb{Q}$ such that

$$\operatorname{ap} \lim_{x \to a} \frac{\mathcal{G}(u(x), v(x))}{|x - a|} = 0.$$

Such a function $v$ is uniquely determined and denoted by $\mathrm{ap}Au(a)$.

**The Sobolev space $\mathcal{Y}_2(U, \mathbb{Q})$.** Suppose $U$ is an open ball in $\mathbb{R}^m$ or the entire space $\mathbb{R}^m$.

1. A function $u \in W^{1,2}(U, \mathbb{R}^n)$ is said to be strictly defined if $u(x) = y$ whenever $x \in U, y \in \mathbb{R}^n$ and

$$\lim_{r \to 0} r^{-m} \int_{\mathbb{B}_r^m(x)} |u(z) - y| \, d\mathcal{L}^m z = 0.$$

2. The space $\mathcal{Y}_2(U, \mathbb{Q})$ consists of those functions $u : U \to \mathbb{Q}$ for which $\xi \circ u \in W^{1,2}\left(U, \mathbb{R}^{PQ}\right)$. We say that $u$ is strictly defined if $\xi \circ u$ is strictly defined.

3. If $u \in \mathcal{Y}_2(U, \mathbb{Q})$, then $u$ is approximately affinely approximatable $\mathcal{L}^m$ almost everywhere on $U$. The Dirichlet integral of $u$ over $U$ is defined by

$$\mathrm{Dir}(u; U) = \int_U |\mathrm{ap}Au(x)|^2 \, d\mathcal{L}^m x.$$

**The Sobolev space $\partial\mathcal{Y}_2(\partial U, \mathbb{Q})$.** Suppose $U \subset \mathbb{R}^m$ is an open ball.

1. The space $\partial\mathcal{Y}_2(\partial U, \mathbb{Q})$ will be the set of those functions $v : \partial U \to \mathbb{Q}$ for which there is a strictly defined $u \in \mathcal{Y}_2(\mathbb{R}^m, \mathbb{Q})$ such that $u(x) = v(x)$ for $\mathcal{H}^{m-1}$ almost all $x \in \partial U$ .

2. If $v \in \partial\mathcal{Y}_2(\partial U, \mathbb{Q})$ we say that $u$ has boundary values $v$ if there is $w \in \mathcal{Y}_2(\mathbb{R}^m, \mathbb{Q})$ which is strictly defined such that $w|_U = u|_U$ and $w|_{\partial U} = v$. We then write $u|_{\partial U} = v$. If $u$ is strictly defined, $v$ agrees with $u$ $\mathcal{H}^{m-1}$ almost everywhere on $\partial U$.

3. One says that $u : U \to \mathbb{Q}$ is Dirichlet minimizing if and only if $u \in \mathcal{Y}_2(U, \mathbb{Q})$ and, assuming $u$ has boundary values $v \in \partial\mathcal{Y}_2(\partial U, \mathbb{Q})$, one has

$$\mathrm{Dir}(u; U) = \inf\{\mathrm{Dir}(w; U) : w \in \mathcal{Y}_2(U, \mathbb{Q}) \text{ has boundary values } v\}.$$

4. For each $v \in \partial\mathcal{Y}_2(\partial U, \mathbb{Q})$, the Dirichlet integral of $v$ over $\partial U$ is defined by

$$\mathrm{dir}(v; \partial U) = \int_{\partial U} |\mathrm{ap}Av(x)|^2 \, d\mathcal{H}^{m-1}x$$

where the affine approximation $\mathrm{ap}Av$ is with respect to $\partial U$.

**Wrapping.** The following procedure of obtaining a multiple-valued function from a single-valued function is called wrapping. For each positive integer $k$, $f \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbb{R}^n)$ having boundary values

$$g \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{R}^n)$$

we construct $F \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbb{Q}_k)$ having boundary values $G \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{Q}_k)$ by setting

$$\begin{aligned} F(\omega) &= \sum\{[[f(z)]] : \omega = z^k\} \text{ for each complex number } \omega \text{ with } 0 < |\omega| < 1, \\ F(0) &= k[[f(0)]], \\ G(\omega) &= \sum\{[[g(z)]] : \omega = z^k\} \text{ for each complex number } \omega \text{ with } |\omega| = 1; \end{aligned}$$

here we are using the usual complex coordinates for $\mathbb{R}^2$.
Since Dirichlet integrals are invariant under two-dimensional conformal mappings, we readily check

$$\mathrm{Dir}(F; \mathbb{B}_1^2(0)) = \mathrm{Dir}(f; \mathbb{B}_1^2(0)); \tag{2}$$

one further readily checks that

$$\mathrm{dir}(G; \partial\mathbb{B}_1^2(0)) = k^{-1}\mathrm{dir}(g; \partial\mathbb{B}_1^2(0)). \tag{3}$$

**Theorem 2.1** ([AF], §2.7). *If $G \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{Q})$, then there exist continuous functions $G^1, G^2, \cdots, G^Q : [0, 2\pi] \to \mathbb{R}^n$ such that for each $0 \le \theta \le 2\pi$,*

$$G(\cos\theta, \sin\theta) = \sum_{i=1}^{Q} [[G^i(\theta)]]. \tag{4}$$

*Renumbering the $\{G^i\}_i$ if necessary, one readily verifies the existence of $J \in \{1, \cdots, Q\}$ and $k_1 \cdots, k_J \in \{1, \cdots, Q\}$ with $\sum_{i=1}^{J} k_i = Q$ such that*

$$G^i(2\pi) = G^{i+1}(0) \text{ for each } i \in \{1, \cdots, Q\} \sim \{k_1, k_1 + k_2, \cdots, Q\} \tag{5}$$

$$G^{k_1+k_2+\cdots+k_\alpha}(2\pi) = G^{k_1+k_2+\cdots+k_{\alpha-1}+1}(0) \text{ for each } \alpha = 1, \cdots, J, \tag{6}$$

*where we use the convention that $k_0 = 0$.*
*One then sets $g_1, \cdots, g_J \in \partial\mathbb{B}_1^2(0) \to \mathbb{R}^n$ by requiring whenever $0 \le \theta \le 2\pi$ and $\alpha \in \{1, \cdots, J\}$*

$$g_\alpha(\cos\theta, \sin\theta) = G^{k_1+k_2+\cdots+k_{\alpha-1}+\beta}(\gamma) \tag{7}$$

*where* $\beta \in \{1, 2, \cdots, k_\alpha\}$ *satisfies* $(\beta - 1)\frac{2\pi}{k_\alpha} \leq \theta \leq \beta\frac{2\pi}{k_\alpha}$ *and* $\gamma = k_\alpha(\theta - (\beta - 1)\frac{2\pi}{k_\alpha})$.
*One verifies that* $g_\alpha \in \mathcal{Y}_2(\partial\mathbb{B}^2_1(0), \mathbb{R}^n)$ *and let* $f_\alpha \in \mathcal{Y}_2(\mathbb{U}^2_1(0), \mathbb{R}^n)$ *be the unique harmonic function having boundary values* $g_\alpha$. *For each* $\alpha = 1, \cdots, J$ *one constructs* $F_\alpha \in \mathcal{Y}_2(\mathbb{U}^2_1(0), \mathbb{Q}_{k_\alpha})$ *from* $f_\alpha$ *by the wrapping procedure and sets*

$$F = F_1 + F_2 + \cdots + F_J \in \mathcal{Y}_2(\mathbb{U}^2_1(0), \mathbb{Q})$$

*and checks that* $F$ *has boundary values* $G$.

**Remark 2.1.** *(1) Given any* $G \in \mathcal{Y}_2(\partial\mathbb{B}^2_1(0), \mathbb{Q})$, *we call the above decomposition* $g_1, \cdots, g_J$ *an un-wrapping. Obviously, un-wrapping is not necessarily unique. But if the boundary function* $G$ *has no branch point, then the un-wrapping is indeed unique (up to reordering).*
*(2) Throughout the paper, we always decompose* $G$ *into* $G^i$*'s in "the most efficient way" so that*

$$G^j(2\pi) \neq G^{k_1+\cdots+k_{\alpha-1}+1}(0), \text{ for } j \in \{k_1 + \cdots + k_{\alpha-1} + 2, \cdots, k_1 + \cdots + k_\alpha - 1\}. \tag{8}$$

**Theorem 2.2** ([AF], §2.8)**.** *For* $m \geq 2$, *suppose* $g \in \mathcal{Y}_2(\partial\mathbb{B}^m_1(0), \mathbb{Q})$ *with* $0 < dir(g; \partial\mathbb{B}^m_1(0))$ *and set*

$$d = dir(g; \partial\mathbb{B}^m_1(0)), H = \int_{x\in\partial\mathbb{B}^m_1(0)} \mathcal{G}(g(x), Q[[0]])^2 d\mathcal{H}^{m-1}x > 0,$$
$$M = d/H, L = 2^{-1}([4M + (m-2)^2]^{1/2} - (m-2)).$$

*For each* $0 < t < \infty$ *we construct* $f_t \in \mathcal{Y}_2(\mathbb{U}^m_1(0), \mathbb{Q})$ *by setting*

$$f_t(x) = \mu(|x|^t)_\sharp g(x/|x|) \text{ for } x \in \mathbb{U}^m_1(0) \sim \{0\}, f_t(0) = Q[[0]].$$

*Then*
*(1)* $Dir(f_t; \mathbb{B}^m_1(0)) = (2t + m - 2)^{-1}(t^2H + d)$.
*(2)* $Dir(f_L; \mathbb{B}^m_1(0)) = HL = \inf\{Dir(f_t; \mathbb{B}^m_1(0)) : 0 < t < \infty\}$.
*(3) Moreover,* $Dir(f_t; \mathbb{B}^2_1(0))$ *assumes its unique minimum in* $(0, \infty)$ *at* $L$.

**Theorem 2.3** ([AF], §2.6)**.** ***Hypotheses:***
*(a)* $0 < r_0 < \infty$, $A \subset \mathbb{R}^m$ *is connected, open, and bounded with* $\mathbb{U}^m_{r_0}(0) \subset A$. $\partial A$ *is an* $m - 1$ *dimensional submanifold of* $\mathbb{R}^m$ *of class 1.*
*(b)* $f : A \to \mathbb{Q}$ *is strictly defined and Dirichlet minimizing.*
*(c)* $D, H, N : (0, r_0) \to \mathbb{R}$ *are defined for* $0 < r < r_0$ *by setting*

$$D(r) = Dir(f; \mathbb{B}^m_r(0))$$
$$H(r) = \int_{\partial\mathbb{B}^m_r(0)} \mathcal{G}(f(x), Q[[0]])^2 d\mathcal{H}^{m-1}x$$
$$N(r) = rD(r)/H(r) \text{ provided } H(r) > 0.$$

*(d)* $\mathcal{N} : A \to \mathbb{R}$ *is defined for* $x \in A$ *by setting*

$$\mathcal{N}(f, x) = \mathcal{N}(x) = \lim_{r\downarrow 0} rDir(f; \mathbb{B}^m_r(x)) / \int_{\partial\mathbb{B}^m_r(x)} \mathcal{G}(f(z), Q[[0]])^2 d\mathcal{H}^{m-1}z$$

*provided this limit exists.*
*(e)* $H(r) > 0$ *for some* $0 < r < r_0$.
***Conclusions.***
*(1)* $N(r)$ *is defined for each* $0 < r < r_0$ *and is nondecreasing.*
*(2)* $\mathcal{N}(0) = \lim_{r\downarrow 0} N(r)$ *exists.*
*(3)* $\mathcal{N}(x)$ *is well defined for each* $x \in A$ *and is upper semicontinuous as a function of* $x$.

**Theorem 2.4** ([AF], §2.13). *Suppose $f \in \mathcal{Y}_2(\mathbb{R}^2, \mathbb{Q})$ is strictly defined and $f|\mathbb{U}_1^2(0)$ is Dirichlet minimizing with $Dir(f; \mathbb{B}_1^2(0)) > 0$. Then*
*(1) Either $f(0) = Q[[0]]$ and $\mathcal{N}(0) \geq 1/Q$ or $f(0) \neq Q[[0]]$ and $\mathcal{N}(0) = 0$.*
*(2) Suppose $f(0) = Q[[0]]$ and $1/2 > r(1) > r(2) > r(3) > \cdots > 0$ with $0 = \lim_{i\to\infty} r(i)$. Then there is a subsequence $i_1, i_2, i_3, \cdots$ of $1, 2, 3, \cdots$ and a function $g : \mathbb{B}_1^2(0) \to \mathbb{Q}$ with the following properties:*
*(a) $g$ is the uniform limit as $k \to \infty$ of the functions*

$$\mu(Dir(f \circ \mu[r(i_k)]; \mathbb{B}_1^2(0))^{-1/2})_\sharp \circ f \circ \mu[r(i_k)]|\mathbb{B}_1^2(0).$$

*(b) $g|\mathbb{U}_1^2(0) \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbb{Q})$ is Dirichlet minimizing with $Dir(g; \mathbb{B}_1^2(0)) = 1$.*
*(c) $\int_{x\in\partial\mathbb{B}_1^2(0)} \mathcal{G}(g(x), Q[[0]])^2 d\mathcal{H}^1 x = 1/\mathcal{N}(0)$.*
*(d) $g(0) = Q[[0]]$ and for each $x \in \mathbb{B}_1^2(0) \sim \{0\}$,*

$$g(x) = \mu(|x|^{\mathcal{N}(0)})_\sharp \circ g(x/|x|).$$

**Theorem 2.5** ([AF],§2.14). *Suppose $f \in \mathcal{Y}_2(\mathbb{U}_1^m(0), \mathbb{Q})$ is strictly defined and Dirichlet minimizing. Define $\sigma : \mathbb{U}_1^m(0) \to \{1, \cdots, Q\}$ by*

$$\sigma(x) = card[spt(f(x))] \text{ for } x \in \mathbb{U}_1^m(0).$$

*Then the set*

$$\Sigma = \mathbb{U}_1^m(0) \cap \{x : \sigma \text{ is not continuous at } x\}$$

*is closed in $\mathbb{U}_1^m(0)$ with Hausdorff dimension not exceeding $m - 2$. In case $m = 2$, $\Sigma$ is countable.*

# 3 Frequency

**Proposition 3.1.** *Suppose $F \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q})$ and denote $F|\partial\mathbb{B}_1^2(0)$ by $G$. If*
*(1) $G \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{Q})$,*
*(2) $F$ is Dirichlet minimizing with $Dir(F; \mathbb{B}_1^2(0)) = 1$,*
*(3) $F(0) = Q[[0]]$ and for each $x \in \mathbb{B}_1^2(0) \sim \{0\}$,*

$$F(x) = \mu(|x|^N)_\sharp \circ G(x/|x|) \text{ for some positive real number } N,$$

*(4) $\int_{x\in\partial\mathbb{B}_1^2(0)} \mathcal{G}(G(x), Q[[0]])^2 d\mathcal{H}^1 x = 1/N$,*
*then $N = p/q$ for some positive integers $p$ and $q$ with $q \leq Q$.*

*Proof.* First of all, by Theorem 2.2,

$$\mathrm{dir}(G; \partial\mathbb{B}_1^2(0)) = N. \tag{9}$$

Apply Theorem 2.1 to function $G$ and let

$$G_\alpha = \sum_{i=k_1+\cdots+k_{\alpha-1}+1}^{k_1+\cdots+k_\alpha} [[G^i]] \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{Q}_{k_\alpha}), \alpha \in \{1, \cdots, J\}.$$

For $\alpha \in \{1, \cdots, J\}$, denote (where $g_\alpha$ comes from $G$ by Theorem 2.1)

$$\begin{aligned} d_\alpha &= \mathrm{dir}(g_\alpha; \partial\mathbb{B}_1^2(0)), \\ H_\alpha &= \int_{x\in\partial\mathbb{B}_1^2(0)} |g_\alpha(x)|^2 d\mathcal{H}^1 x, \\ \tilde{d}_\alpha &= \mathrm{dir}(G_\alpha; \partial\mathbb{B}_1^2(0)), \\ \tilde{H}_\alpha &= \int_{x\in\partial\mathbb{B}_1^2(0)} \mathcal{G}(G_\alpha(x), k_\alpha[[0]])^2 d\mathcal{H}^1 x. \end{aligned}$$

We readily check

$$d_\alpha = k_\alpha \tilde{d}_\alpha, H_\alpha = \frac{\tilde{H}_\alpha}{k_\alpha}. \tag{10}$$

For each $\alpha \in \{1, \cdots, J\}$, let $f_\alpha$ be the homogeneous degree $N \cdot k_\alpha$ extension of $g_\alpha$. We use wrapping procedure to construct $F_\alpha \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q}_{k_\alpha})$ from $g_\alpha$ and $f_\alpha$ and let $\tilde{F} = \sum_{i=1}^{J} F_i \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q})$. Apparently, $F|\partial\mathbb{B}_1^2(0) = \tilde{F}|\partial\mathbb{B}_1^2(0) = G$. For each $\alpha \in \{1, \cdots, J\}$, by Theorem 2.2 we have

$$\begin{aligned}
\mathrm{Dir}(F_\alpha; \mathbb{B}_1^2(0)) &= \mathrm{Dir}(f_\alpha; \mathbb{B}_1^2(0)) \\
&= (2N \cdot k_\alpha)^{-1}((N \cdot k_\alpha)^2 H_\alpha + d_\alpha) \\
&= (2N \cdot k_\alpha)^{-1}((N \cdot k_\alpha)^2 \frac{\tilde{H}_\alpha}{k_\alpha} + k_\alpha \tilde{d}_\alpha) \\
&= \frac{N^2 \tilde{H}_\alpha + \tilde{d}_\alpha}{2N}.
\end{aligned} \tag{11}$$

Therefore

$$\mathrm{Dir}(\tilde{F}; \mathbb{B}_1^2(0)) = \sum_{\alpha=1}^{J} \mathrm{Dir}(F_\alpha; \mathbb{B}_1^2(0)) = \sum_{\alpha=1}^{J} \frac{N^2 \tilde{H}_\alpha + \tilde{d}_\alpha}{2N} = \frac{N^2 H + d}{2N} = 1, \tag{12}$$

where the last equality comes from $H = \int_{\partial\mathbb{B}_1^2(0)} \mathcal{G}(G(x), Q[[0]])^2 d\mathcal{H}^1 = \frac{1}{N}$ and $d = dir(G; \partial\mathbb{B}_1^2(0)) = N$. Since $F$ is Dirichlet minimizing and $\mathrm{Dir}(F; \mathbb{B}_1^2(0)) = \mathrm{Dir}(\tilde{F}; \mathbb{B}_1^2(0)) = 1$, each $f_\alpha$ is harmonic. Therefore, for each $\alpha \in \{1, \cdots, J\}$ such that $g_\alpha$ is not constant zero, $N \cdot k_\alpha$ must be a positive integer. Since $G$ is not a constant function, there must be at least one such $\alpha$. Taking the sum of those $N \cdot k_\alpha$ such that $g_\alpha$ is not constant zero, we get

$$Nq = p \text{ for some positive integers } p, q \text{ with } q \leq Q.$$

□

*Proof of Theorem 1.2.* It suffices to prove the theorem for the origin. We also observe that in spirit of Theorem 2.4 (1), we may assume $f(0) = Q[[0]]$ and $\mathrm{Dir}(f; \mathbb{B}_1^2(0)) > 0$. Applying Theorem 2.4 to function $f$ yields a function $g \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q})$ of homogeneous degree $\mathcal{N}(0)$. Moreover, we have

$$g|\partial\mathbb{B}_1^2(0) \in \mathcal{Y}_2(\partial\mathbb{B}_1^2(0), \mathbb{Q})$$

Proposition 3.1 yields that

$$\mathcal{N}(0) = p/q, \text{ for some positive integers } p, q \text{ with } q \leq Q.$$

□

**Theorem 3.1.** *Under the same assumptions of Proposition 3.1, moreover, if $K \leq J$ is a non-negative integer such that*

$$k_1 = \cdots = k_K = 1, g_1 = \cdots = g_K \equiv 0,$$

*and*

$$g_i \textit{ not constant zero}, i = K+1, \cdots J.$$

*Moreover, we set*

$$N \cdot k_i = l_i \textit{ for positive integer } l_i, i \in \{K+1, \cdots, J\}.$$

*Then each $l_i$ and $k_i$ are relatively prime, for $i = K+1, \cdots, J$.*

*Proof.* If not, without worrying about abusing the notation, let's assume that $k_1$ and $l_1$ are not relatively prime. Therefore, there is a positive integer $s > 1$ which divides both $k_1$ and $l_1$. Using the same notation as before, let's consider the harmonic function $f_1$ of homogeneous degree $N \cdot k_1 = l_1$. We can write $f_1$ as

$$f_1(r, \theta) = r^{l_1} g_1(\theta).$$

From basic facts about homogeneous harmonic functions, we know each component of $g_1$ is a linear combination of functions $\cos(l_1\theta)$ and $\sin(l_1\theta)$. Hence

$$g_1(\theta) = g_1(\theta + \frac{2\pi}{s}),$$

specifically, $g_1(0) = g_1(\frac{2\pi}{s})$. Therefore, $G^1(0) = G^{k_1/s}(2\pi)$, which is in contradiction to our assumption (8) for the decomposition. □

**Corollary 3.1.** *Under the same assumptions, we have*

$$k_{K+1} = \cdots = k_J, l_{K+1} = \cdots = l_J. \tag{13}$$

*Proof.* This is because from $\frac{l_i}{k_i} = \frac{l_j}{k_j}$, we have

$$l_i k_j = l_j k_i.$$

Therefore $k_j$ divides $l_j k_i$. But since we know $k_j$ and $l_j$ are relatively prime, $k_j$ divides $k_i$. Similarly, $k_i$ divides $k_j$. Hence $k_i = k_j$, which also implies $l_i = l_j$. □

**Remark 3.1.** *(1) In case $N = 1/Q$, then obviously $K = 0$. Moreover, $J = 1, k_1 = Q$. This is because, from Theorem 3.1,*

$$N \cdot k_i = l_i \text{ for positive integer } l_i, i \in \{K+1, \cdots, Q\}$$

*i.e. either $K = Q$ or $k_{K+1} = Q$. If $K = Q$ then the original function is zero, in contradiction to the assumption $N = 1/Q$. If $k_{K+1} = Q$, then $K$ has to be zero and $J = 1, k_1 = Q$.*
*(2) But conversely, if $J = 1, k_1 = Q$, we do not necessarily have $N = 1/Q$. For example, the function $z^{2/3}$ gives a $3-$valued function with $J = 1$ and frequency $2/3$ at the origin.*
*(3) If $N$ is a positive integer, it is easy to see that $k_1 = k_2 = \cdots = k_J = 1$ and $J = Q$. This is because $N \cdot k_i = l_i$ and $k_i$ is relatively prime with $l_i$ for $i = K+1, \cdots, J$.*

# 4 Classification of branch points

Since interior branch points are isolated for two-dimensional multiple-valued Dirichlet minimizers, we will assume that the origin is the only branch point of a 2-valued Dirichlet minimizer $f : \mathbb{B}_1^2(0) \to \mathbf{Q}_2(\mathbb{R}^n)$. Our results can be easily extended to $Q-$valued cases, for $Q \geq 3$. Furthermore, we may assume $f(0) = 2[[0]]$ because if $f(0) = 2[[a]]$ for some non-zeron $a \in \mathbb{R}^n$, we can consider the function $f(-)a$.

**Definition 4.1.** *For a strictly defined, Dirichlet minimizing function $f \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ with $f(0) = 2[[0]]$ the only branch point, we define*

$$C(r) = (\partial\mathbb{B}_r^2(0) \times \mathbb{R}^n) \cap \{(x, y) : y \in spt(f(x))\}, 0 < r \leq 1.$$

$$M(r) = \text{number of components of } C(r), 0 < r \leq 1.$$

**Remark 4.1.** *(1) It is easy to see from [AF] §2.14 that $C(r)$ is an analytic curve having either one or two components.*
*(2) From the interior regularity and the assumption that the origin is the only branch point, we conclude that $M(r)$ is a constant function.*

**Definition 4.2.** *For a strictly defined, Dirichlet minimizing function* $f \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ *with* $f(0) = 2[[0]]$ *the only branch point, we call the origin a type-one branch point if* $M(r) \equiv 1$*, otherwise, a type-two branch point.*

**Theorem 4.1.** *For a strictly defined, Dirichlet minimizing function* $f \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ *with* $f(0) = 2[[0]]$ *the only branch point of type-one, there is a harmonic function* $g : \mathbb{U}_1^2(0) \to \mathbb{R}^n$ *such that*

$$f(r, \theta) = [[g(r^{1/2}, \frac{\theta}{2})]] + [[g(r^{1/2}, \frac{\theta}{2} + \pi)]], 0 \leq r < 1, 0 \leq \theta \leq 2\pi. \tag{14}$$

*Proof.* First of all, we observe that for a.e. $0 < r < 1$, $f|\partial\mathbb{B}_r^2(0) \in \mathcal{Y}_2(\partial\mathbb{B}_r^2(0), \mathbf{Q}_2(\mathbb{R}^n))$. For these $r$, let $h_r : \mathbb{S}^1 \to \mathbb{R}^n$ be the un-wrapping function associated with $f|\partial\mathbb{B}_r^2(0)$. It is unique because there is no branch point on $\partial\mathbb{B}_r^2(0)$. Moreover,

$$f(r, \theta) = [[h_r(\frac{\theta}{2})]] + [[h_r(\frac{\theta}{2} + \pi)]]. \tag{15}$$

We define $g : \mathbb{U}_1^2(0) \to \mathbb{R}^n$ by

$$g(r, \theta) = h_{r^2}(\theta), 0 < r < 1, 0 \leq \theta \leq 2\pi,$$

$$g(0) = 0.$$

The continuity of $f$ on $\mathbb{U}_1^2(0)$ guarantees that $g$ is well-defined and continuous on $\mathbb{U}_1^2(0)$. It is also easy to check that (14) holds.
Since Dirichlet integrals are invariant under two-dimensional conformal mappings, we conclude

$$\mathrm{Dir}(f; \mathbb{B}_1^2(0)) = \mathrm{Dir}(g; \mathbb{B}_1^2(0)). \tag{16}$$

To show that $g$ is harmonic, we take any function $\phi \in \mathcal{Y}_2(\mathbb{B}_1^2(0); \mathbb{R}^n)$ such that $\phi = g$ on $\mathbb{S}^1$. We construct $\Phi \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ by wrapping $\phi$ and readily check $\Phi = f$ on $\mathbb{S}^1$, and

$$\mathrm{Dir}(\phi; \mathbb{B}_1^2(0)) = \mathrm{Dir}(\Phi; \mathbb{B}_1^2(0)) \leq \mathrm{Dir}(f; \mathbb{B}_1^2(0)) = \mathrm{Dir}(g; \mathbb{B}_1^2(0)), \tag{17}$$

which completes the proof. □

**Theorem 4.2.** *For a strictly defined, Dirichlet minimizing function* $f \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ *with* $f(0) = 2[[0]]$ *the only branch point of type-two, there are two harmonic functions* $g_1, g_2 : \mathbb{U}_1^2(0) \to \mathbb{R}^n$*, such that*

$$f(x) = [[g_1(x)]] + [[g_2(x)]], x \in \mathbb{U}_1^2(0), \tag{18}$$

$$g_1(0) = g_2(0) = 0, g_1(x) \neq g_2(x), \forall x \in \mathbb{U}_1^2(0) \sim \{0\}. \tag{19}$$

*Proof.* Since $C(r)$ has two components for every $0 < r < 1$, we conclude

$$L_0^* = (\mathbb{U}_1^2(0) \sim \{0\}) \times \mathbb{R}^n \cap \{(x, y) : y \in \mathrm{spt}(f(x))\}$$

has two components (see e.g. [AF] §2.16). Suppose that $L_0^*$ as above has two components $L_1^*$ and $L_2^*$. Then there are harmonic functions $g_1, g_2 \in \mathcal{Y}_2(\mathbb{U}_1^2(0); \mathbb{R}^n)$ such that $L_1^* = \{(x, g_1(x)) : x \in \mathbb{U}_1^2(0) \sim \{0\}\}$, $L_2^* = \{(x, g_2(x)) : x \in \mathbb{U}_1^2(0) \sim \{0\}\}$, and $f = [[g_1]] + [[g_2]]$. The theorem is proved once we notice the fact that isolated points are removable singularities of bounded harmonic functions (see e.g. [HP]§2.1) □

Next we relate the characterization of branch points with frequency.

**Theorem 4.3.** *For a strictly defined, Dirichlet minimizing function $f \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbf{Q}_2(\mathbb{R}^n))$ with $f(0) = 2[[0]]$ the only branch point of type-two, then the frequency at the origin is a positive integer. More precisely, let $f = [[g_1]] + [[g_2]]$ as Theorem 4.2, then*

$$\mathcal{N}(f, 0) = \min\{\mathcal{N}(g_1, 0), \mathcal{N}(g_2, 0)\}. \tag{20}$$

*If the origin is of type-one, with $f(r, \theta) = [[g(r^{1/2}, \frac{\theta}{2})]] + [[g(r^{1/2}, \frac{\theta}{2} + \pi)]]$ as in Theorem 4.1, then*

$$\mathcal{N}(f, 0) = \mathcal{N}(g, 0)/2. \tag{21}$$

*Proof.* It is easy to check that (we abbreviate $\mathrm{Dir}(f, \mathbb{B}_r^m(0))$ by $D(f, r)$)

$$D(f, r) = D(g_1, r) + D(g_2, r), H(f, r) = H(g_1, r) + H(g_2, r). \tag{22}$$

Also by the properties of two-dimensional harmonic functions, we have

$$D(g_1, r) \text{ is of order } r^{2\mathcal{N}(g_1,0)}, D(g_2, r) \text{ is of order } r^{2\mathcal{N}(g_2,0)},$$

$$H(g_1, r) \text{ is of order } r^{2\mathcal{N}(g_1,0)+1}, H(g_2, r) \text{ is of order } r^{2\mathcal{N}(g_2,0)+1}.$$

Noticing that

$$\mathcal{N}(f, 0) = \lim_{r \to 0} \frac{r(D(g_1, r) + D(g_2, r))}{H(g_1, r) + H(g_2, r)},$$

we conclude $\mathcal{N}(f, 0) = \min\{\mathcal{N}(g_1, 0), \mathcal{N}(g_2, 0)\}$.
Now we suppose $f(0) = 2[[0]]$ is a type-one branch point, we readily check

$$D(f, r) = D(g, r^{1/2}), H(f, r) = 2H(g, r^{1/2})r^{1/2}. \tag{23}$$

Therefore,

$$\mathcal{N}(f, 0) = \lim_{r \to 0} \frac{rD(g, r^{1/2})}{2H(g, r^{1/2})r^{1/2}} = \frac{1}{2} \lim_{r \to 0} \frac{r^{1/2}D(g, r^{1/2})}{H(g, r^{1/2})} = \frac{\mathcal{N}(g, 0)}{2}.$$

□

**Remark 4.2.** *(1) From this theorem, type-two branch point corresponds to integer frequency. Type-one branch point can have either integer or non-integer as its frequency. Here are two examples: for $f(z) = z^{1/2}$, the origin is a type-one branch point with frequency $1/2$; for $f(z) = [[z + z^{3/2}]] + [[z - z^{3/2}]]$, the origin is also a type-one branch point but with frequency $1$.*
*(2) However, for a two-dimensional homogeneous multiple-valued Dirichlet minimizing function $f$, if $f(0) = 2[[0]]$ is the only branch point, then $\mathcal{N}(f, 0)$ is an integer if and only if the origin is a type-two branch point, $\mathcal{N}(f, 0) = p/2$ for an odd integer $p$ if and only if the origin is a type-one branch point. To prove this, it suffices to show that if the origin is the only branch point with type-one, then $\mathcal{N}(f, 0) = p/2$ for an odd integer $p$. This is because if $\mathcal{N}(f, 0)$ is an integer, by Remark 3.1(3), $J = 2$, a contradiction.*

Similarly, for $Q \geq 3$, we have

**Theorem 4.4.** *Suppose $f \in \mathcal{Y}_2(\mathbb{B}_1^2(0), \mathbb{Q})$ is strictly defined and Dirichlet minimizing with $f(0) = Q[[0]]$ the only branch point. Then there are harmonic functions $f_i, i = 1, \cdots, J : \mathbb{B}_1^2(0) \to \mathbb{R}^n$ such that*

$$f(z) = \sum_{i=1}^{J} \{[[f_i(w)]] : w^{k_i} = z\}, \textit{for each complex number } z \textit{ with } 0 < |z| \leq 1,$$

$$f(0) = \sum_{i=1}^{J} k_i [[f_i(0)]],$$

*where $J \in \{1, \cdots, Q\}, k_i \in \{1, \cdots, Q\}$, and $\sum_{i=1}^{J} k_i = Q$.*

**Theorem 4.5.** *Under the same assumptions as above, we have*

$$\mathcal{N}(f,0)=\min\{\frac{\mathcal{N}(f_1,0)}{k_1},\cdots,\frac{\mathcal{N}(f_J,0)}{k_J}\}. \tag{24}$$

**Remark 4.3.** *This actually gives another way to prove Theorem 1.2.*

# 5 Uniqueness of blowing-up functions

As before, we only prove the case $Q=2$ and the result is easily extended to the general cases $Q\geq 3$.

**Theorem 5.1.** *For a strictly defined, Dirichlet minimizing function $f\in\mathcal{Y}_2(\mathbb{B}_1^2(0),\mathbf{Q}_2(\mathbb{R}^n))$,*
*(1) Suppose $f(0)=2[[0]]$ is a branch point of type-one with $g$ as in Theorem 4.1. Let $G:\mathbb{B}_1^2(0)\to\mathbb{R}^n$ be the blowing-up function of $g$ at the origin, then the blowing-up function of $f$ at the origin is unique and given by*

$$(r,\theta)\to[[G(r^{1/2},\frac{\theta}{2})]]+[[G(r^{1/2},\frac{\theta}{2}+\pi)]].$$

*(2) Suppose $f(0)=2[[0]]$ is a type-two branch point with $g_1,g_2$ as in Theorem 4.2. Let $G_1,G_2:\mathbb{B}_1^2(0)\to\mathbb{R}^n$ be the blowing-up functions of $g_1,g_2$ at the origin respectively. If $\mathcal{N}(g_1,0)<\mathcal{N}(g_2,0)$, then the blowing-up function of $f$ at the origin is given by*

$$[[G_1]]+[[0]].$$

*If $\mathcal{N}(g_1,0)>\mathcal{N}(g_2,0)$, then the blowing-up function of $f$ at the origin is given by*

$$[[G_2]]+[[0]].$$

*If $\mathcal{N}(g_1,0)=\mathcal{N}(g_2,0)$, then the blowing-up function of $f$ at the origin is given by*

$$[[\frac{G_1}{\sqrt{1+M}}]]+[[G_2\sqrt{\frac{M}{1+M}}]],$$

*where $M=\lim_{r\to 0}D(f_2,r)/D(f_1,r)<\infty$.*

*Proof.* In the first case, since $f(r,\theta)=[[g(r^{1/2},\frac{\theta}{2})]]+[[g(r^{1/2},\frac{\theta}{2}+\pi)]]$, we have

$$\begin{aligned}F(r,\theta)&=\lim_{s\to 0}\frac{f(sr,\theta)}{D(f,s)^{1/2}}\\&=\lim_{s\to 0}\frac{[[g((sr)^{1/2},\frac{\theta}{2})]]+[[g((sr)^{1/2},\frac{\theta}{2}+\pi)]]}{D(g,s^{1/2})^{1/2}}\\&=\lim_{s\to 0}[[\frac{g((sr)^{1/2},\frac{\theta}{2})}{D(g,s^{1/2})^{1/2}}]]+\lim_{s\to 0}[[\frac{g((sr)^{1/2},\frac{\theta}{2}+\pi)}{D(g,s^{1/2})^{1/2}}]]\\&=[[G(r^{1/2},\frac{\theta}{2})]]+[[G(r^{1/2},\frac{\theta}{2}+\pi)]].\end{aligned} \tag{25}$$

The proof of the second case is similar to the proof of the first one. □

**Remark 5.1.** *For $Q\geq 3$, and $f(0)=Q[[0]]$ is an isolated branch point, following the same argument above combined with Theorem 4.4, we also conclude that blowing-up functions at the origin are unique.*

## 6 Boundary regularity

Let

$$\mathbb{B}_\rho^{m,+}(0) := \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m, x_m \geq 0, x_1^2 + x_2^2 + \cdots + x_m^2 \leq \rho\},$$

whose boundary consists of two parts: $\mathbb{S}_\rho^{m,+}(0)$ the spherical part and $\mathbb{I}_\rho$ the flat part. Since the regularity is a local property, we may just consider the case when the domain is the upper half unit ball, namely, $f \in \mathcal{Y}_2(\mathbb{B}_1^{m,+}(0), \mathbb{Q})$.

*Proof of Theorem 1.3.* Without loss of generality, we may assume that $\xi \circ f(0) = 0$ (otherwise, we can consider the function $(\xi \circ f)(x) - (\xi \circ f)(0)$). Let $M$ be the energy of $\xi \circ f$ in $\mathbb{B}_1^{2,+}(0)$ which is finite because $\xi$ is Lipschitzian. Applying Courant-Lebesgue lemma to $\xi \circ f$, we get a sequence $r_i \downarrow 0$, such that

$$|\xi \circ f(x) - \xi \circ f(y)| \leq (4\pi M)^{1/2} (\log \frac{1}{r_i})^{-1/2}, \forall x, y \in \mathbb{S}_{r_i}^{2,+}(0). \tag{26}$$

Combined with the assumption that $\xi \circ f|\mathbb{I}_1$ is continuous with $\xi \circ f(0) = 0$, we have

$$\sup_{x \in \partial \mathbb{B}_{r_i}^{2,+}(0)} |\xi \circ f(x)| \to 0, \text{ as } i \to \infty.$$

Applying the maximum principle [ZW] (it is very straightforward to show that the maximum principle works for the function $\xi \circ f$ too if $f$ is Dirichlet minimizing), we get

$$\sup_{x \in \mathbb{B}_r^{2,+}(0)} |\xi \circ f(x)| \to 0, \text{ as } r \to 0,$$

which proves the theorem. □

This boundary continuity can be improved near the non-branch point on the boundary.

**Theorem 6.1.** *Under the same assumption as above, if the origin is not a branched point, i.e. there are $r > 0$, $J \in \{1, \cdots, Q\}$, $k_i \in \{1, \cdots, Q\}$ with $\sum_{i=1}^J k_i = Q$, $f|\mathbb{I}_r = \sum_{i=1}^J k_i[[g_i]]$ for continuous $g_i \in \partial\mathcal{Y}_2(\mathbb{I}_r, \mathbb{R}^n)$ and $g_i(x) \neq g_j(x), \forall x \in \mathbb{I}_r, i \neq j$. Then there is $\delta > 0$ small enough such that in $\mathbb{B}_\delta^{2,+}(0)$,*

$$f = \sum_{i=1}^J k_i[[f_i]]$$

*where each $f_i$ is a single-valued harmonic function and $f_i(x) \neq f_j(x), \forall x \in \mathbb{B}_\delta^{2,+}(0), i \neq j$. In particular, there are no branch points converging to a non-branch point on the boundary.*

*Proof.* It is an easy consequence of the boundary regularity and the assumption that the origin is a non-branch point on the boundary. □

**Remark 6.1.** *It is still not clear whether a sequence of interior branch points can converge to a branch point on the boundary.*

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