Math 355.1 Instructor: Wei Zhu Solution to Exam #1

1. (1) False, because the determinant of these two matrices are different.
   (2) False, because any orthogonal set of nonzero vectors must be linear independent. In \( \mathbb{R}^2 \), there is no linear independent of three vectors.
   (3) True. If \( A \) is singular, then \( \det(A) = 0 \). Hence for the characteristic polynomial \( \det(A - \lambda I) \), if you let \( \lambda = 0 \), it gives you zero. Therefore 0 is an eigenvalue of \( A \).

2. \( \Gamma([1 0]^T) = [1 2 0]^T, \Gamma([0 1]^T) = [1 - 1 2]^T \)

   Therefore
   \[
   A = \begin{pmatrix}
   1 & 1 \\
   2 & -1 \\
   0 & 2
   \end{pmatrix}
   \]

   It is also easy to see the matrix resulted from changing the bases:

   \[
   P = \begin{pmatrix}
   1 & 1 \\
   0 & 1
   \end{pmatrix},
   S = \begin{pmatrix}
   1 & 1 & 0 \\
   0 & 1 & 1 \\
   0 & 0 & 1
   \end{pmatrix}
   \]

   By using multiple-augmented matrix method, we are able to see

   \[
   S^{-1} = \begin{pmatrix}
   1 & -1 & 1 \\
   0 & 1 & -1 \\
   0 & 0 & 1
   \end{pmatrix}
   \]

   Therefore,

   \[
   A' = S^{-1}AP = \begin{pmatrix}
   -1 & 3 \\
   2 & -1 \\
   0 & 2
   \end{pmatrix}
   \]

   By definition of \( \Gamma \), \( \Gamma([1 1]^T) = [2 1 2]^T \).

   On the other hand, the coordinate vector of \([1 1]^T\) under the basis \( B' \) is \([0 1]^T\). Hence the coordinate of vector \( \Gamma([1 1]^T) \) under the basis \( C' \) should be

   \[
   A' \begin{pmatrix}
   0 \\
   1
   \end{pmatrix} = \begin{pmatrix}
   3 \\
   -1 \\
   2
   \end{pmatrix}
   \]

   The vector in \( \mathbb{R}^3 \) which has coordinate vector \([3 - 1 2]^T\) under the basis \( C' \) is exactly \([2 1 2]^T\). This finishes the problem.

3. (1)

   \[
   \det(A - \lambda I) = \det\left(\begin{pmatrix}
   1 - \lambda & 0 & 0 \\
   2 & -5 - \lambda & -6 \\
   -2 & 3 & 4 - \lambda
   \end{pmatrix}\right) = -(\lambda - 1)^2(\lambda + 2)
   \]
So there are two distinct eigenvalues, $\lambda_1 = 1, m_1 = 2$ and $\lambda_2 = -2, m_2 = 1$.

(2) For $\lambda_1$,
\[
A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}
\]
We form the augmented matrix
\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -6 & -6 & 0 \\ -2 & 3 & 3 & 0 \end{pmatrix}
\]
and put it into Gauss reduced form to get
\[
\begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
Hence the general form of the eigenvalues of $\lambda_1$ are
\[
\begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}
\]
It tells that $\mu_1 = 1$.

Similarly, for $\lambda_2$,
\[
A - \lambda_2 I = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}
\]
Doing the same thing, you will find that the general form of the eigenvalues of $\lambda_2$ are
\[
\begin{pmatrix} 0 \\ -2k \\ k \end{pmatrix}
\]
Hence $\mu_2 = 1$.

4. It is easy to find the eigenvalues of $A$, which are 2 and 3. Take the corresponding eigenvectors of them to be:
\[
\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\]
We let
\[
Q = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},
Q^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}
\]
It is easy to check that

\[ Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \]

5. For \( \lambda_1 \),

\[ A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix} \]

\[ E_1^1 = \{ k \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \}, \dim(E_1^1) = 1 < m_1, \]

keep doing,

\[ (A - \lambda_1 I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 18 & 18 \\ 0 & -9 & -9 \end{pmatrix} \]

\[ E_1^2 = \{ \begin{pmatrix} l \\ -k \\ k \end{pmatrix} = l \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \}, \dim(E_1^1) = 2 = m_1 \]

Therefore \( d_1 = 1, d_2 = 1 \). Make the diagram:

\[ v_2 \quad v_1 \]

Choose \( v_1 \) to be in \( E_1^2 \) but not in \( E_1^1 \). We take

\[ v_1 = [1 \ 0 \ 0]^T \]

Then \( v_2 = (A - \lambda_1 I)v_1 = [0 \ 2 \ -2]^T \).

As for \( \lambda_2 \),

\[ A - \lambda_2 I = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix} \]

\[ E_{-2}^1 = \{ k \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \}, \dim(E_{-2}^1) = 1 = m_2 \]

So \( d_1 = 1 \). We make the diagram

\[ v_1 \]

Take \( v_1 \) to be

\[ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \]
So the transition matrix should be
\[
Q = \begin{pmatrix}
0 & 1 & 0 \\
2 & 0 & -2 \\
-2 & 0 & 1
\end{pmatrix}
\]
\[
Q^{-1} = \begin{pmatrix}
0 & -1/2 & -1 \\
1 & 0 & 0 \\
0 & -1 & -1
\end{pmatrix}
\]

It is easy to compute that
\[
Q^{-1}AQ = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}
\]

6. (1) \(J\) should have two Jordan blocks. The sizes of them should be two and one respectively.
\[
J = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \text{ or } \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

(2) No we can not tell the Jordan form. \(J\) should have two Jordan blocks, the sizes of them can either be 2\&2 or 1\&3. We have two possibilities, up to reordering the Jordan blocks:
\[
J = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \text{ or } \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

7. Just follow the process, you will get
\[
u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} -2/9 \\ 5/9 \\ 1 \\ -4/9 \end{pmatrix}, u_3 = \begin{pmatrix} 1/7 \\ -5/14 \\ 5/14 \\ 2/7 \end{pmatrix}
\]