Math 355.1 Instructor: Wei Zhu **Solution to Exam #1**

1.(1) False, because the determinant of these two matrices are different.
(2) False, because any orthogonal set of nonzero vectors must be linear independent. In $\mathbb{R}^2$, there is no linear independent of three vectors.
(3) True. If A is singular, then $\det(A) = 0$. Hence for the characteristic polynomial $\det(A - \lambda I)$, if you let $\lambda = 0$, it gives you zero. Therefore 0 is an eigenvalue of A.

2.

$$\Gamma([1\ 0]^T) = [1\ 2\ 0]^T, \Gamma([0\ 1]^T) = [1\ -1\ 2]^T$$

Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 2 \end{pmatrix}$$

It is also easy to see the matrix resulted from changing the bases:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

By using multiple-augmented matrix method, we are able to see

$$S^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$A' = S^{-1}AP = \begin{pmatrix} -1 & 3 \\ 2 & -1 \\ 0 & 2 \end{pmatrix}$$

By definition of Γ, $\Gamma([1\ 1]^T) = [2\ 1\ 2]^T$.
On the other hand, the coordinate vector of $[1\ 1]^T$ under the basis B' is $[0\ 1]^T$.
Hence the coordinate of vector $\Gamma([1\ 1]^T)$ under the basis C' should be

$$A' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

The vector in $\mathbb{R}^3$ which has coordinate vector $[3\ -1\ 2]^T$ under the basis C' is exactly $[2\ 1\ 2]^T$. This finishes the problem.

3. (1)

$$det(A - \lambda I) = det(\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{pmatrix}) = -(\lambda - 1)^2(\lambda + 2)$$

So there are two distinct eigenvalues, $\lambda_1 = 1, m_1 = 2$ and $\lambda_2 = -2, m_2 = 1$.
(2) For λ_1,

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}$$

We form the augmented matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -6 & -6 & 0 \\ -2 & 3 & 3 & 0 \end{pmatrix}$$

and put it into Gauss reduced form to get

$$\begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the general form of the eigenvalues of λ_1 are

$$\begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}$$

It tells that $\mu_1 = 1$.
Similarly, for λ_2,

$$A - \lambda_2 I = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}$$

Doing the same thing, you will find that the general form of the eigenvalues of λ_2 are

$$\begin{pmatrix} 0 \\ -2k \\ k \end{pmatrix}$$

Hence $\mu_2 = 1$.

4. It is easy to find the eigenvalues of A, which are 2 and 3. Take the corresponding eigenvectors of them to be:

$$\vec{x_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{x_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

We let

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},$$

$$Q^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

It is easy to check that

$$Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

5. For λ_1,

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}$$

$$E_1^1 = \{k \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\}, \dim(E_1^1) = 1 < m_1,$$

keep doing,

$$(A - \lambda_1 I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 18 & 18 \\ 0 & -9 & -9 \end{pmatrix}$$

$$E_1^2 = \{\begin{pmatrix} l \\ -k \\ k \end{pmatrix} = l \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\}, \dim(E_1^1) = 2 = m_1$$

Therefore $d_1 = 1, d_2 = 1$. Make the diagram:

$$v_2$$

$$v_1$$

Choose v_1 to be in E_1^2 but not in E_1^1. We take

$$v_1 = [1\ 0\ 0]^T$$

Then $v_2 = (A - \lambda_1 I)v_1 = [0\ 2\ -2]^T$.
As for λ_2,

$$A - \lambda_2 I = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}$$

$$E_{-2}^1 = \{k \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\}, \dim(E_{-2}^1) = 1 = m_2$$

So $d_1 = 1$. We make the diagram

$$v_1$$

Take v_1 to be

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

So the transition matrix should be

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 0 & -1/2 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

It is easy to compute that

$$Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

6.(1) J should have two Jordan blocks. The sizes of them should be two and one respectively.

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) No we can not tell the Jordan form. J should have two Jordan blocks, the sizes of them can either be 2&2 or 1&3. We have two possibilities, up to reordering the Jordan blocks:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. Just follow the process, you will get

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} -2/9 \\ 5/9 \\ 1 \\ -4/9 \end{pmatrix}, u_3 = \begin{pmatrix} 1/7 \\ -5/14 \\ 5/14 \\ 2/7 \end{pmatrix}$$