**Math 355.1 Instructor:** Wei Zhu **Solution to Exam #2**

1. (1) False, because dimension of $\mathbb{R}^3$ is three.
(2) False, because dimension of $\mathbb{R}^2$ should equal to dimension of null space + dimension of image space, therefore dimension of image space can not be three.
(3) True.

2. (a) Since $\vec{v} = 7(1) + (-3)(t) + 4(t^2)$,

$$C_B(\vec{v}) = \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}.$$

(b)

$$\begin{aligned} 1 + t &= 1(1) + 1(t) + 0(t^2), \\ 1 - t &= 1(1) + (-1)(t) + 0(t^2), \\ 1 + t^2 &= 1(1) + (0)(t) + 1(t^2), \end{aligned}$$

Therefore,

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using the method of multiple-augmented matrix, we can find the inverse of $M$,

$$M^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) By Theorem 5.43,

$$C_{B'}(\vec{v}) = M^{-1}C_B(\vec{v}) = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

3. Let

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 1 \end{pmatrix}$$

Doing elementary row operations on $A$ gives us

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The first three columns of $G$ are leading columns. Therefore

$$\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \}$$

is a basis of $V_0$.

4. The equation $x_1 + x_2 = x_3 + x_4$ is equivalent to $x_1 + x_2 - x_3 - x_4 = 0$, which has three free variables. Let's assign

$$x_4 = k, x_3 = l, x_2 = m,$$

which gives $x_1 = k + l - m$.
Hence the general vectors in $V_0$ are

$$\begin{pmatrix} k+l-m \\ m \\ l \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + l \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + m \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore

$$\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \}$$

spans $V_0$. It is also quite easy to show it is also linear independent. Hence it is a basis of $V_0$.
Now we form a matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Doing elementary row operations to $A$ gives us

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

whose first, second and forth columns are leading columns. Therefore

$$\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

is a basis for $V_0$.

5. Let

$$A_1 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -4 \end{pmatrix}$$

Doing elementary row operations to $A_1$ and $A_2$ gives us

$$R_1 = R_2 = \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & -7/5 \end{pmatrix}$$

Hence $S_1$ and $S_2$ span the same subspace of $\mathbb{R}^3$.

6. (a) For any two polynomials $f, g$ in $\mathcal{V}$, $\Gamma(f+g) = (f+g)\cdot t = f\cdot t + g\cdot t = \Gamma(f)+\Gamma(g)$. For any real number $\alpha$, $\Gamma(\alpha f) = (\alpha f)\cdot t = \alpha(f\cdot t) = \alpha\Gamma(f)$. Hence by definition, $\Gamma$ is a linear transformation from $\mathcal{V}$ to $\mathcal{W}$.
(b) $\Gamma(f) = f\cdot t$ equals zero polynomials in $\mathcal{W}$ if and only if $f$ equals zero polynomials in $\mathcal{V}$. Hence the null space of $\Gamma$ is $\{\vec{0}\}$., which has dimension zero. By the equality

$$dim(\mathcal{V}) = dim(\text{null space of }\Gamma) + dim(\text{image space of }\Gamma),$$

we know that the dimension of the image space of $\Gamma$ is the same as dimension of $\mathcal{V}$, which is 2.
(c)

$$\Gamma(1) = 1\cdot t = t = 0(1) + 1(t) + 0(t^2),$$
$$\Gamma(t) = t\cdot t = t^2 = 0(1) + 0(t) + 1(t^2)$$

Therefore the matrix representation of $\Gamma$ is the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(d) The matrix $P$ relating $B$ with $B'$ is found by the following way:

$$1+t = 1(1) + 1(t), 1-t = 1(1) + (-1)(t),$$

hence

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Similary, for the matrix $S$ relating $C$ with $C'$,

$$1 = 1(1) + 0(t) + 0(t^2), 1+t = 1(1) + 1(t) + 0(t^2), 1+t^2 = 1(1) + 0(t) + 1(t^2),$$

hence

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By the method of multiple-augmented matrix, we find the inverse of $S$:

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By Theorem 6.17,

$$A' = S^{-1}AP = \begin{pmatrix} -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$