# An energy reducing flow for multiple-valued functions *

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**Abstract**

For a positive integer $Q$, a $Q-$valued function $f$ between $\mathbb{R}^m$ and $\mathbb{R}^n$ associates to each $x \in \mathbb{R}^m$ an unordered $Q-$tuple of points in $\mathbb{R}^n$. The Dirichlet energy of such a function is described in [AF]. By the method of time discretization, we construct an energy reducing $Q-$valued function flow. This flow is Hölder continuous with respect to the $L^2$ norm. Some questions concerning regularity and branching behaviors are also addressed.

## 1 Introduction

This work was originally motivated by the paper [CX], in which X. Cheng constructed a mass reducing flow for integral currents. The idea of his construction comes from Horihata and Kikuchi's paper [HK] for nonlinear parabolic equations. More specifically, let $T_0$ be an integral current, and let $h > 0$ be given. He defines a step approximation sequence $\{T_h^k\}$ of integral currents with the same boundary as that of $T_0$ by choosing $T_h^k$ such that $T_h^k$ minimizes the functional

$$G(T) = G(T_h^{k-1}, T, h) = M(T)^2 + \frac{F_k(T_h^{k-1} - T)^2}{h}, \tag{1}$$

where $T$ is an integral current with $\partial T = \partial T_0$. Then he constructed a $k+1$ dimensional current $S_h$ by "connecting" those $T_h^k$. Finally he takes a weak limit $S$ of $S_h$ as $h \to 0$, slices $S$ with respect to $t$ to get an integral current at time $t$. The flow is Hölder continuous under the flat norm and reduces the mass of the initial integral current while keeping the boundary fixed. Later on, this same time discretization process was used in Haga, Hoshino and Kikuchi's paper [HHK] to construct a harmonic map flow. This gives an alternative proof of the classical result due to J. Eells, Jr. and J. H. Sampson [ES] and to R. S. Hamilton [HR].

Our work is trying to construct a multiple-valued function flow by a similar time discretization method. One obvious obstacle here is that we do not have differential equations for multiple-valued functions. Therefore a lot of PDE methods cannot be applied. Another thing that stands in the way is due to the lack of fundamental algebraic operations, for example, addition for multiple-valued functions. Hence we cannot use linear interpolation to connect those multiple-valued functions at different stages. All of these problems will be handled with great care.

Here are the main theorems of this paper ($\mathcal{Y}_2$ is the Sobolev space $W^{1,2}$ for the version involving multiple-valued functions. See the preliminaries below or [AF]):

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**Theorem 1.1.** *Assume that* $f_0 \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{Q})$, *strictly defined and* $Dir(f_0; \mathbb{B}_1^m(0)) < \infty, ||\xi \circ f_0||_{L^2} < \infty$. *There exists* $F \in \mathcal{Y}_2([0,\infty) \times \mathbb{B}_1^m(0), \mathbb{Q})$ *such that*

$$F(0, x) = f_0(x), x \in \mathbb{B}_1^m(0),$$

$$F(t, x) = f_0(x), t \in [0, \infty), x \in \partial\mathbb{B}_1^m(0).$$

*If we denote* $F_t(\cdot) : \mathbb{B}_1^m(0) \to \mathbb{Q},\ F_t(x) = F(t, x)$, *for any* $t \in [0, \infty)$, *then for* $\mathcal{L}^1$ *almost every* $t > 0$,

$$Dir(F_t; \mathbb{B}_1^m(0)) \le Dir(f_0; \mathbb{B}_1^m(0)).$$

*Moreover the flow is Hölder continuous w.r.t the* $L^2$ *norm, i.e.,*

$$||\xi \circ F_t - \xi \circ F_s||_{L^2} \le \sqrt{s-t}\sqrt{Dir(f_0; \mathbb{B}_1^m(0))}$$

*for* $0 \le t < s$ *for* $\mathcal{L}^1$ *almost every* $t, s$.

**Theorem 1.2.** *Under the same assumption as above, suppose* $f_0 = [[g]] + [[-g]] \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{Q}_2(\mathbb{R}))$, *with nonnegative function* $g \in C^\infty(\mathbb{B}_1^m(0), \mathbb{R})$. *Then* $\eta(f_h^k) \equiv 0$ *for* $k = 0, 1, 2, \cdots$. *Moreover, if we denote*

$$f_h^k(x) = [[\tilde{f_h^k}(x)]] + [[-\tilde{f_h^k}(x)]], \tilde{f_h^k}(x) \ge 0, k = 0, 1, 2, \cdots.$$

*then each* $\tilde{f_h^k} \in C^\infty(\mathbb{B}_1^m(0), \mathbb{R})$ *and satisfies the following PDEs:*

$$\frac{\tilde{f_h^k} - \tilde{f_h^{k-1}}}{h/2^k} = \Delta \tilde{f_h^k}$$

*If* $g$ *is not identically zero, then* $\tilde{f_h^k}(x) > 0, \forall x \in \mathbb{U}_1^m(0)$, $k = 1, 2, \cdots$, *i.e., the flow has no branch point.*

There remains many interesting questions about the nature of this flow. One of them, for example, is that whether the energy of this flow satisfies

$$\mathrm{Dir}(F_s; \mathbb{B}_1^m(0)) \le \mathrm{Dir}(F_t; \mathbb{B}_1^m(0)), \forall 0 \le t \le s.$$

Another one is the optimal regularity of such flows. For energy minimizing functions, Almgren [AF] has proven the partial interior regularity.

# 2 Preliminaries

The theory of multiple-valued functions was developed in [AF] and later on further studied ,e.g, in [GJ],[LC],[MP]. It is the most natural framework for the regularity theory in geometric measure theory and promises a lot of future development and applications in other fields. Here we introduce the basic notations and facts of multiple-valued functions. The reader is referred to [AF] for more details. We also use standard terminology in geometric measure theory, all of which can be found on page 669-671 of the treatise *Geometric Measure Theory* by H. Federer [FH].

The space $\mathbb{Q} = \mathbb{Q}_Q(\mathbb{R}^n)$ consists of all the unordered $Q$ points in $\mathbb{R}^n$, denoted by $\sum_{i=1}^{Q}[[p_i]], p_i \in \mathbb{R}^n$. We let $\mathrm{spt}(\sum_{i=1}^{Q}[[p_i]]) = \cup_{i=1}^{Q}\{p_i\}$.

We define a metric on $\mathbb{Q}$

$$\mathcal{G} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$$

by setting for $p_1, \cdots, p_Q, q_1, \cdots, q_Q \in \mathbb{R}^n$,

$$\mathcal{G}(\sum_i [[p_i]], \sum_i [[q_i]]) = \inf_\sigma \{(\sum_i |p_i - q_{\sigma(i)}|^2)^{1/2} : \sigma \text{ is a permutation of } \{1, \cdots, Q\}\}. \quad (2)$$

We let $|\sum_{i=1}^Q [[p_i]]|^2 = \sum_{i=1}^Q |p_i|^2 = \mathcal{G}^2(\sum_{i=1}^Q [[p_i]], Q[[0]])$.

The average function $\eta : \mathbb{Q} \to \mathbb{R}^n$ is defined to be

$$\eta(\sum_{i=1}^Q [[p_i]]) = \sum_{i=1}^Q p_i/Q \quad (3)$$

We define

$$\zeta : \mathbb{O}^*(n,1) \times \mathbb{Q} \to \mathbb{R}^Q \cap \{s : s_1 \le s_2 \le s_3 \le \cdots \le s_Q\}$$

for $\pi \in \mathbb{O}^*(n,1), p \in \mathbb{Q}$ by requiring $-\infty < s_1 \le s_2 \le s_3 \le \cdots \le s_Q < \infty$ and $\pi_\sharp p = \sum_{i=1}^Q [[s_i]]$. One can easily check that Lip($\zeta(\pi,\cdot)$)=1 for each $\pi \in \mathbb{O}^*(n,1)$.

**Theorem 2.1** ([AF],§1.2)**.** *There exist $\Pi_1, \Pi_2, \cdots, \Pi_p \in \mathbb{O}^*(n,1)$ ($P$ is a positive integer depending on $n, Q$ defined in [AF], §1.1, $P \ge n$) such that*
*(a) $\Pi_i(x) = x_i$ for each $i = 1, \cdots, n$ and each $x \in \mathbb{R}^n$.*
*(b) Lip($\xi_0$) = 1.*
*(c) $\xi : \mathbb{Q} \to \mathbb{Q}^*$ is a bilipschitzian homeomorphism where*

$$\xi = \zeta(\Pi_1, \cdot) \bowtie \cdots \bowtie \zeta(\Pi_p, \cdot) : \mathbb{Q} \to \mathbb{R}^{PQ}, \mathbb{Q}^* = \xi(\mathbb{Q})$$

*and*

$$\xi_0 = \zeta(\Pi_1, \cdot) \bowtie \cdots \bowtie \zeta(\Pi_n, \cdot) : \mathbb{Q} \to \mathbb{R}^{nQ}.$$

*(Here we use the notation that whenever $f : A \to B$ and $g : A \to C$, we define*

$$f \bowtie g : A \to B \times C, (f \bowtie g)(a) = (f(a), g(a)), a \in A.)$$

**Remark 2.1.** *Brian White showed there is a modified bilipschitzian correspondance $\xi : \mathbb{Q} \to \mathbb{Q}^* \subset \mathbb{R}^{PQ}$ such that for every $p \in \mathbb{Q}$, $p$ has a small neighbourhood in $\mathbb{Q}$ such that $\xi$ is an equidistance map over the neighbourhood. The modification is to choose the orthogonal projections $\Pi_1, \cdots, \Pi_P$ in [AF] as complete sets of coordinate projections corresponding to distinct orthonormal coordinate systems for $\mathbb{R}^n$ and to compose the resulting map $\xi$ there with proper scaling to get such a $\xi$. It has some other useful properties that we will mention later. Moreover, we will use the modified $\xi$ throughout the rest of this paper.*

**Theorem 2.2** ([AF],§1.3)**.** *There exists an explicitly constructable, piecewise linear function*

$$\rho : \mathbb{R}^{PQ} \to \mathbb{R}^{PQ}$$

*such that Lip($\rho$) $< \infty$, $\rho(\mathbb{R}^{PQ}) \subset \mathbb{Q}^*$, and $\rho(x) = x$ for each $x \in \mathbb{Q}^*$.*

**Definition 2.1** ([AF])**.** *(a) $f$ is a Q-valued function on some subset $U$ of $\mathbb{R}^m$ if it is a map*

$$f : U \subset \mathbb{R}^m \to \mathbb{Q}$$

*(b) For a given smooth, compact embedded manifold $N$ in $\mathbb{R}^n$, define*

$$\underline{Q}(N) = \{\sum_{i=1}^Q [[p_i]], p_i \in N, i = 1, \cdots, Q\}$$

*(c) $f$ is a Q-valued map from some subset $U$ of $\mathbb{R}^m$ into $N$ if it is a map*

$$f : U \subset \mathbb{R}^m \to \underline{Q}(N) \subset \mathbb{Q}$$

*(d) Similarly we can define $\underline{Q}(V)$ for any vector space $V$.*

**Definition 2.2** ([AF]). *(a) $f$ is called a Q-valued Lipschitz function(map) if there is a constant $C > 0$ such that*

$$\mathcal{G}(f(x), f(y)) \leq C|x - y|, \quad x, y \in U.$$

*(b) $f$ is called affine if there are $A_1, \cdots, A_Q$ where each $A_i$ is an affine map from $\mathbb{R}^m$ to $\mathbb{R}^n$, such that*

$$f(x) = \sum_{i=1}^{Q} [[A_i(x)]].$$

*(c) $f$ is called affinely approximatable at $x_0$ if there are affine maps $A_1, \cdots, A_Q$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that*

$$\lim_{|x-x_0|\to 0} \frac{\mathcal{G}(f(x), \sum_{i=1}^{Q} [[A_i(x)]])}{|x - x_0|} = 0. \tag{4}$$

*(d) $f$ is strongly affinely approximatable at $x_0$ if (c) holds for $f$ at $x_0$ and $A_i = A_j$ if $A_i(x_0) = A_j(x_0)$.*

**Remark 2.2.** *(1) From [AF],§1.4, if $f$ is a Q-valued Lipschitz function, then it is strongly affinely approximatable almost everywhere over its domain.*
*(2) If $f$ is affinely approximatable at $x_0$ with $\sum_{i=1}^{Q} [[A_i]]$ as its affine approximation, then obviously $f(x_0) = \sum_{i=1}^{Q} [[A_i(x_0)]]$ and $A_i(x) = A_i(x_0) + L_i(x - x_0)$ with $L_i \in$ $Hom(\mathbb{R}^m, \mathbb{R}^n)$.*

**Definition 2.3** ([AF]). *If $f$ is affinely approximatable at $x_0$, then*
*(a) $\sum_{i=1}^{Q} [[L_i]] \in \underline{Q}(Hom(\mathbb{R}^m, \mathbb{R}^n))$, denoted by $Df(x_0)$ is defined as the differential of $f$ at $x_0$. We let $|Df(x_0)|^2 = \sum_{i=1}^{Q} |L_i|^2$, where $|L|$ is the Euclidean norm of the matrix associated with any $L \in Hom(\mathbb{R}^m, \mathbb{R}^n)$.*
*(b) $\sum_{i=1}^{Q} [[L_i(v)]]$ is defined as the derivative of $f$ at $x_0$ in the direction $v$ and is denoted by $D_v f \in \mathbb{Q}$. Let $|D_v f(x_0)|^2 = \sum_{i=1}^{Q} |L_i(v)|^2$.*

**Remark 2.3.** *The map $\xi$ mentioned before has the following properties*

$$|\xi \circ f| = |f|, |D_v(\xi \circ f)| = |D_v f| \tag{5}$$

**Definition 2.4.** *Suppose $A \subset \mathbb{R}^m$ is bounded and open and that $\partial A$ is an $m - 1$ dimensional submanifold of $\mathbb{R}^m$ of class 1. Whenever $V$ is a Euclidean vector space, $\mathcal{Y}_2(\mathbb{R}^m, V)$ and $\mathcal{Y}_2(A, V)$ are the real vector spaces of square summable functions whose distribution first derivatives are also square summable. $\partial\mathcal{Y}_2(\partial A, V)$ is the real vector space of all $\mathcal{H}^{m-1}$ measurable function $f : \partial A \to V$ such that*

$$\int_{\partial A} |f|^2 d\mathcal{H}^{m-1} + \int_{z\in\partial A} |z|^{-m} \int_{x\in\partial A} |f(x + z) - f(z)|^2 d\mathcal{H}^{m-1} x d\mathcal{H}^{m-1} z < \infty \tag{6}$$

**Definition 2.5** ([AF]). *(a) We define*

$$\mathcal{Y}_2(\mathbb{R}^m, \mathbb{Q}) \text{ [resp. } \mathcal{Y}_2(A, \mathbb{Q})]$$

*to be the space of all functions $f : \mathbb{R}^m \to \mathbb{Q}$[resp. $f : A \to \mathbb{Q}$] such that $\xi \circ f \in \mathcal{Y}_2(\mathbb{R}^m, \mathbb{R}^{PQ})$[resp. $\xi \circ f \in \mathcal{Y}_2(A, \mathbb{R}^{PQ})$]. We also define*

$$\partial\mathcal{Y}_2(\partial A, \mathbb{Q})$$

*as the space of all functions $g : \partial A \to \mathbb{Q}$ such that $\xi \circ g \in \partial\mathcal{Y}_2(\partial A, \mathbb{R}^{PQ})$.*
*(b) For each $f \in \mathcal{Y}_2(A, \mathbb{Q})$ and each $\mathcal{L}^m$ measurable set $K \subset \mathbb{R}^m$ which is $\mathcal{L}^m$ almost a subset of $A$, we define*

$$Dir(f; K) = Dir(\xi_0 \circ f; K).$$

*For each $g \in \partial\mathcal{Y}_2(\partial A, \mathbb{Q})$ and $\mathcal{H}^{m-1}$ measurable set $L \subset \mathbb{R}^m$ which is $\mathcal{H}^{m-1}$ almost a subset of $\partial A$, we set*

$$dir(g; L) = dir(\xi_0 \circ g; L).$$

**Remark 2.4.** *From Remark 2.3, we have*

$$Dir(f;K) = Dir(\xi \circ f;K). \tag{7}$$

**Definition 2.6** ([AF])**.** *One says that $f : A \to \mathbb{Q}$ is Dirichlet minimizing if and only if $f \in \mathcal{Y}_2(A,\mathbb{Q})$ and, assuming $f$ has boundary values $g \in \partial\mathcal{Y}_2(\partial A,\mathbb{Q})$, one has*

$$Dir(f;A) = \inf\{Dir(h;A) : h \in \mathcal{Y}_2(A,\mathbb{Q}) \text{ has boundary values } g\}.$$

# 3 Construction of the flow

We assume that $f_0 \in \mathcal{Y}_2(\mathbb{B}_1^m(0),\mathbb{Q})$, strictly defined and

$$Dir(f_0;\mathbb{B}_1^m(0)) < \infty, ||\xi \circ f_0||_{L^2} < \infty.$$

Define

$$\mathcal{M} = \{u \in \mathcal{Y}_2(\mathbb{B}_1^m(0),\mathbb{Q}), \text{strictly defined}, u = f_0 \text{ on } \partial\mathbb{B}_1^m(0)\}.$$

Given an $h > 0$, we define inductively, for $k = 1, 2, 3, \cdots$, a multiple-valued function $f_h^k$ such that $f_h^0 = f_0$ and $f_h^k$ minimizes the functional

$$G(g) = G(f_h^{k-1}, g, h) = Dir(g;\mathbb{B}_1^m(0)) + \frac{1}{h/2^k}||\xi \circ f_h^{k-1} - \xi \circ g||_{L^2}^2$$

where $g \in \mathcal{M}$.

**Proposition 3.1.** *$f_h^k$ is defined for each positive integer $k$ and each $h > 0$.*

*Proof.* Let

$$L_k = \inf\{G(f_h^{k-1}, g, h) : g \in \mathcal{M}\}, k = 1, 2, 3, \cdots$$

Since $G(f_h^0, f_0, h) = Dir(f_0;\mathbb{B}_1^m(0)) < \infty$, $L_1 < \infty$. Let $\{g_i\} \subset \mathcal{M}$ be a sequence such that

$$\lim_{i\to\infty} G(f_h^0, g_i, h) = L_1.$$

It is easy to see

$$Dir(g_i;\mathbb{B}_1^m(0)) \le G(f_h^0, g_i, h) \le \sup_i G(f_h^0, g_i, h) < \infty. \tag{8}$$

By Theorem (10) in [AF], $\S A.1.2$, there is a subsequence, still denoted as $g_i$ such that

$$g_i \rightharpoonup g \in \mathcal{Y}_2.$$

We have

$$\begin{aligned}\int_{\mathbb{B}_1^m(0)} |\xi \circ f_h^0 - \xi \circ g|^2 dx &= \int_{\mathbb{B}_1^m(0)} \lim_{i\to\infty} |\xi \circ f_h^0 - \xi \circ g_i|^2 dx \\ &\le \liminf_{i\to\infty} \int_{\mathbb{B}_1^m(0)} |\xi \circ f_h^0 - \xi \circ g_i|^2 dx\end{aligned} \tag{9}$$

and

$$Dir(g;\mathbb{B}_1^m(0)) \le \liminf_{i\to\infty} Dir(g_i;\mathbb{B}_1^m(0)) < \infty \tag{10}$$

with $g = f_h^0$ on $\partial\mathbb{B}_1^m(0)$. By redefining $g$ at some points if necessary, we may assume $g \in \mathcal{M}$. Therefore

$$\begin{aligned} G(f_h^0, g, h) &= Dir(g; \mathbb{B}_1^m(0)) + \frac{\int_{\mathbb{B}_1^m(0)} |\xi \circ f_h^0 - \xi \circ g|^2 dx}{h/2} \\ &\le \liminf_{i\to\infty} Dir(g_i; \mathbb{B}_1^m(0)) + \frac{\liminf_{i\to\infty} \int_{\mathbb{B}_1^m(0)} |\xi \circ f_h^0 - \xi \circ g_i|^2 dx}{h/2} \\ &\le \liminf_{i\to\infty} G(f_h^0, g_i, h) = L_1. \end{aligned} \tag{11}$$

Hence $G(f_h^0, g, h) = L_1$. We just let $f_h^1$ to be $g$. Repeat the process, we are done. ☐

**Proposition 3.2.**

$$Dir(f_h^k; \mathbb{B}_1^m(0)) \le Dir(f_h^{k-1}; \mathbb{B}_1^m(0)) \le Dir(f_0; \mathbb{B}_1^m(0)). \tag{12}$$

$$||\xi \circ f_h^{k-1} - \xi \circ f_h^k||_{L^2}^2 \le \frac{h}{2^k}(Dir(f_h^{k-1}; \mathbb{B}_1^m(0)) - Dir(f_h^k; \mathbb{B}_1^m(0))). \tag{13}$$

*Proof.* The proof is quite straightforward once we notice that

$$G(f_h^{k-1}, f_h^k, h) \le G(f_h^{k-1}, f_h^{k-1}, h) = Dir(f_h^{k-1}; \mathbb{B}_1^m(0)).$$

☐

Fix $h > 0$, we will construct a multiple-valued function $F_h$ in $[0,\infty) \times \mathbb{B}_1^m(0)$ such that $F_h(0,x) = f_0(x), x \in \mathbb{B}_1^m(0)$, and $F_h(t,x) = f_0(x), t \in [0,\infty), x \in \partial\mathbb{B}_1^m(0)$.
When $t \in [(i-1)h, ih]$, $i = 1,2,3,\cdots$, define

$$F_h(t,x) := f_h^{i-1}(x), \text{ if } t \in [(i-1)h, ih - \frac{h}{2^i}] \tag{14}$$

$$F_h(t,x) := \xi^{-1} \circ \rho \circ [\frac{ih-t}{h/2^i}\xi \circ f_h^{i-1} + \frac{t-(ih-\frac{h}{2^i})}{h/2^i}\xi \circ f_h^i] \text{ if } t \in [ih - \frac{h}{2^i}, ih]. \tag{15}$$

It is easy to see that the function $F_h$ is well-defined and satisfies the boundary condition by using the fact that $\rho \circ \xi = \xi$. Also $F_h \in \mathcal{Y}_2([0,\infty) \times \mathbb{B}_1^m(0), \mathbb{Q})$.
Denoting by $F_k$ the function $F_{1/2^k}$. We will show that for any $T > 0$,

$$\sup_k Dir(F_k; [0,T] \times \mathbb{B}_1^m(0)) < \infty. \tag{16}$$

Choose a positive integer $N$ such that $(N-1)h \le T < Nh$, where $h = 1/2^k$.
Since $F_h(t,x) = f_h^{i-1}(x), t \in [(i-1)h, ih - \frac{h}{2^i}]$,

$$\begin{aligned} &\text{Energy of } F_h \text{ over } [(i-1)h, ih - \frac{h}{2^i}] \times \mathbb{B}_1^m(0) \\ &= (h - \frac{h}{2^i}) \text{ Energy of } f_h^{i-1} \\ &\le h \text{ Energy of } f_h^{i-1} \\ &\le h \text{ Energy of } f_0. \end{aligned} \tag{17}$$

Summing them up, we get

$$\text{Energy of } F_h \text{ over } (\cup_i[(i-1)h, ih - \frac{h}{2^i}] \times \mathbb{B}_1^m(0)) \cap [0,T] \times \mathbb{B}_1^m(0) \le T \cdot \text{ Energy of } f_0. \tag{18}$$

As for the case when $t \in [ih - \frac{h}{2^i}, ih], i = 1, 2, 3, \cdots,$

$$
\begin{aligned}
|\frac{\partial(\xi \circ F_h)}{\partial t}|^2 &\leq (\mathrm{Lip}\rho)^2 |\frac{-1}{h/2^i}\xi \circ f_h^{i-1} + \frac{1}{h/2^i}\xi \circ f_h^i|^2 \\
&= \frac{(\mathrm{Lip}\rho)^2}{(h/2^i)^2}|\xi \circ f_h^{i-1} - \xi \circ f_h^i|^2.
\end{aligned}
\tag{19}
$$

Integrating of that gives

$$
\begin{aligned}
&\int_{ih-h/2^i}^{ih} dt \int_{\mathbb{B}_1^m(0)} |\frac{\partial(\xi \circ F_h)}{\partial t}|^2 dx \\
&\leq \int_{ih-h/2^i}^{ih} dt \int_{\mathbb{B}_1^m(0)} \frac{(\mathrm{Lip}\rho)^2}{(h/2^i)^2}|\xi \circ f_h^{i-1} - \xi \circ f_h^i|^2 dx \\
&= (\mathrm{Lip}\rho)^2 \frac{\int_{\mathbb{B}_1^m(0)} |\xi \circ f_h^{i-1} - \xi \circ f_h^i|^2 dx}{h/2^i} \\
&\leq (\mathrm{Lip}\rho)^2 (Dir(f_h^{i-1}; \mathbb{B}_1^m(0)) - Dir(f_h^i; \mathbb{B}_1^m(0))).
\end{aligned}
\tag{20}
$$

As for $\frac{\partial(\xi \circ F_h)}{\partial x}$, we have

$$
\begin{aligned}
|\frac{\partial(\xi \circ F_h)}{\partial x}|^2 \leq\ & (\mathrm{Lip}\rho)^2 |\frac{ih - t}{h/2^i}\frac{\partial(\xi \circ f_h^{i-1})}{\partial x} + \frac{t - (ih - h/2^i)}{h/2^i}\frac{\partial(\xi \circ f_h^i)}{\partial x}|^2 \\
&\leq 2(\mathrm{Lip}\rho)^2[(\frac{ih - t}{h/2^i})^2|\frac{\partial(\xi \circ f_h^{i-1})}{\partial x}|^2 + (\frac{t - (ih - h/2^i)}{h/2^i})^2|\frac{\partial(\xi \circ f_h^i)}{\partial x}|^2]
\end{aligned}
\tag{21}
$$

Integrating of that gives

$$
\begin{aligned}
&\int_{ih-h/2^i}^{ih} dt \int_{\mathbb{B}_1^m(0)} |\frac{\partial(\xi \circ F_h)}{\partial x}|^2 dx \\
&\leq 2(\mathrm{Lip}\rho)^2 \int_{ih-\frac{h}{2^i}}^{ih} dt \int_{\mathbb{B}_1^m(0)} [(\frac{ih - t}{h/2^i})^2|\frac{\partial(\xi \circ f_h^{i-1})}{\partial x}|^2 + (\frac{t - (ih - \frac{h}{2^i})}{h/2^i})^2|\frac{\partial(\xi \circ f_h^i)}{\partial x}|^2]dx \\
&= 2(\mathrm{Lip}\rho)^2(\frac{h}{3 \times 2^i}) \int_{\mathbb{B}_1^m(0)} |\frac{\partial(\xi \circ f_h^{i-1})}{\partial x}|^2 + |\frac{\partial(\xi \circ f_h^i)}{\partial x}|^2 dx \\
&= 2(\mathrm{Lip}\rho)^2(\frac{h}{3 \times 2^i})(Dir(f_h^{i-1}; \mathbb{B}_1^m(0)) + Dir(f_h^i; \mathbb{B}_1^m(0))) \\
&\leq 2(\mathrm{Lip}\rho)^2(\frac{h}{3 \times 2^i})2Dir(f_0; \mathbb{B}_1^m(0))
\end{aligned}
\tag{22}
$$

Hence

$$
\begin{aligned}
&\text{Energy of } F_h \text{ over } (\cup_i[ih - h/2^i, ih] \times \mathbb{B}_1^m(0)) \cap [0, T] \times \mathbb{B}_1^m(0) \\
&\leq \sum_{i=1}^{N}[(\mathrm{Lip}\rho)^2(Dir(f_h^{i-1}; \mathbb{B}_1^m(0)) - Dir(f_h^i; \mathbb{B}_1^m(0))) + 2(\mathrm{Lip}\rho)^2(\frac{h}{3 \times 2^i})2Dir(f_0; \mathbb{B}_1^m(0))] \\
&\leq (\mathrm{Lip}\rho)^2 Dir(f_0; \mathbb{B}_1^m(0)) + \frac{4}{3}(\mathrm{Lip}\rho)^2 Dir(f_0; \mathbb{B}_1^m(0))(T + h) \\
&\leq (\mathrm{Lip}\rho)^2 Dir(f_0; \mathbb{B}_1^m(0)) + \frac{4}{3}(\mathrm{Lip}\rho)^2 Dir(f_0; \mathbb{B}_1^m(0))(T + 1) < \infty
\end{aligned}
\tag{23}
$$

In summary, we have

$$Dir(F_h; [0,T] \times \mathbb{B}_1^m) < \infty \text{ uniformly for } h = 1/2^k. \tag{24}$$

Using Theorem (10) in [AF], $\S A.1.2$, we have

**Theorem 3.1.** *There exists a subsequence of $F_k$ converging weakly in $\mathcal{Y}_2$ to a multiple-valued function $F \in \mathcal{Y}_2([0,\infty) \times \mathbb{B}_1^m(0), \mathbb{Q})$ such that*

$$F(0,x) = f_0(x), x \in \mathbb{B}_1^m(0),$$

$$F(t,x) = f_0(x), t \in [0,\infty), x \in \partial\mathbb{B}_1^m(0).$$

**Definition 3.1.** *Denote*

$$F_t(\cdot) : \mathbb{B}_1^m(0) \to \mathbb{Q},\ F_t(x) = F(t,x),$$

*for any $t \in [0,\infty)$.*

We are aiming to prove the following two theorems in the rest of this section which completes the proof of Theorem 1.1.

**Theorem 3.2.** *For $\mathcal{L}^1$ almost every $t > 0$,*

$$Dir(F_t; \mathbb{B}_1^m(0)) \le Dir(f_0; \mathbb{B}_1^m(0)). \tag{25}$$

**Theorem 3.3.** *The flow is Hölder continuous with respect to the $L^2$ norm, i.e.,*

$$||\xi \circ F_t - \xi \circ F_s||_{L^2} \le \sqrt{s-t}\sqrt{Dir(f_0; \mathbb{B}_1^m(0))} \tag{26}$$

*for $0 \le t < s$ for $\mathcal{L}^1$ almost every $t, s$.*

**Lemma 3.1.** *Suppose $A_i, i = 1, 2, \cdots$ are measurable sets in $\mathbb{R}$, $\sum_{i=1}^\infty \mathcal{L}^1(A_i) < \infty$, then*

$$\mathcal{L}^1(\overline{\lim} A_k) = 0.$$

*Proof.* See e.g [JF], page 58. □

**Lemma 3.2.** *Suppose $u_i, u \in \mathcal{Y}_2([0,\infty) \times \mathbb{B}_1^m(0), \mathbb{R}^{PQ})$, $u_i \rightharpoonup u$ in $\mathcal{Y}_2$, $\sup_i ||u_i||_{\mathcal{Y}_2} < \infty$. Then for $\mathcal{L}^1$ almost every $t > 0$, there is a subsequence $i_k$, such that $u_{i_k}(t,\cdot), u(t,\cdot) \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{R}^{PQ})$ and*

$$u_{i_k}(t,\cdot) \rightharpoonup u(t,\cdot) \text{ in } \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{R}^{PQ}).$$

*Proof.* It suffices to prove in the case when $u = 0$ and the domain is $[0,1] \times \mathbb{B}_1^m(0)$.
Suppose

$$||u_i||_{\mathcal{Y}_2}^2 = \int_0^1 dt \int_{\mathbb{B}_1^m(0)} |u_i|^2 + |\nabla u_i|^2 dx \le M < \infty.$$

Since $|\nabla u_i(t,\cdot)| \le |\nabla u_i|$,

$$\begin{aligned}\int_0^1 \underline{\lim} ||u_i(t,\cdot)||_{\mathcal{Y}_2}^2 dt &\le \underline{\lim} \int_0^1 ||u_i(t,\cdot)||_{\mathcal{Y}_2}^2 dt \\ &\le \underline{\lim} \int_0^1 dt \int_{\mathbb{B}_1^m(0)} |u_i|^2 + |\nabla u_i|^2 dx \le M < \infty.\end{aligned} \tag{27}$$

Therefore, for $\mathcal{L}^1$ almost every $t > 0$, $\underline{\lim} ||u_i(t,\cdot)||_{\mathcal{Y}_2}^2 < \infty$. By definition of $\liminf$, for such $t$, there is a subsequence $i'$(which may depend on $t$), such that

$$\lim_{i' \to \infty} ||u_{i'}(t,\cdot)||_{\mathcal{Y}_2}^2 < \infty.$$

By compactness theorem, there is a subsequence $i''$ such that

$$u_{i''}(t,\cdot) \rightharpoonup h_t, \text{ for some } h_t \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{R}^{PQ})$$

Hence $u_{i''}(t,\cdot) \to h_t$ in $\mathcal{L}^2$. We will show that $h_t = 0$.

$$\begin{aligned}\int_0^1 \underline{\lim}||u_{i''}(t,\cdot)||_{L^2}^2 dt &\le \underline{\lim}\int_0^1 ||u_{i''}(t,\cdot)||_{L^2}^2 dt \\ &= \underline{\lim}||u_{i''}||_{L^2}^2 \to 0\end{aligned} \tag{28}$$

where the last limit comes from the assumption that $u_{i''} \rightharpoonup 0$ in $\mathcal{Y}_2$. So for $\mathcal{L}^1$ almost every $t > 0$, $\underline{\lim}||u_{i''}(t,\cdot)||_{L^2}^2 = 0$. For such $t$, there is a subsequence $i'''$ such that

$$u_{i'''}(t,\cdot) \to 0 \text{ in } L^2.$$

This proves the lemma. □

*Proof of Theorem 3.2.* Define

$$A_k = \cup_{i=1}^{\infty}[ih - h/2^i, ih], h = 1/2^k \tag{29}$$

$$\mathcal{L}^1(A_k) = \sum_{i=1}^{\infty} \frac{h}{2^i} = h = 1/2^k \tag{30}$$

$$\sum_{k=1}^{\infty} \mathcal{L}^1(A_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \tag{31}$$

By Lemma 3.1, $\mathcal{L}^1(\overline{\lim}A_k) = 0$. Applying Lemma 3.2 to $F_k$, there is a subset $B \subset [0,\infty)$, with $\mathcal{L}^1(B) = 0$, such that for any $t \notin B$, there is a subsequence (still denoted as $k$) such that

$$F_k(t,\cdot) \rightharpoonup F_t \text{ in } \mathcal{Y}_2.$$

Notice that

$$(\overline{\lim}A_k)^c = \underline{\lim}(A_k^c) = \{t: \text{ there exists } n_0, \text{ when } k \ge n_0, x \in A_k^c\}$$

When $t \notin B \cup \overline{\lim}A_k$, after finite steps, $t \notin A_k$ for any $k$, i.e., after finite steps,

$$F_k(t,x) = f_h^{l-1}(x), \text{ for } h = 1/2^k, (l-1)h \le t < lh.$$

Therefore

$$Dir(F_t; \mathbb{B}_1^m(0)) \le \underline{\lim}Dir(F_k(t,\cdot); \mathbb{B}_1^m(0)) = \underline{\lim}Dir(f_h^{l-1}; \mathbb{B}_1^m(0)) \le Dir(f_0; \mathbb{B}_1^m(0))$$

for any $t \notin B \cup \overline{\lim}A_k$. □

*Proof of Theorem 3.3.* Take $t, s \notin B \cup \overline{\lim}A_k$, $t < s$, there is a subsequence (still denoted as $k$) such that

$$F_k(t,\cdot) \to F_t, F_k(s,\cdot) \to F_s \text{ in } L^2.$$

After some finite steps,

$$F_k(t,x) = f_h^{l-1}(x), \text{ for } h = 1/2^k, (l-1)h \le t < lh \tag{32}$$

$$F_k(s,x) = f_h^{l'-1}(x), \text{ for } h = 1/2^k, (l'-1)h \le s < l'h. \tag{33}$$

Therefore $||\xi \circ F_k(t,\cdot) - \xi \circ F_k(s,\cdot)||_{L^2}^2 = ||\xi \circ f_h^{l-1} - \xi \circ f_h^{l'-1}||_{L^2}^2$ when $k$ is big enough.
Using the basic inequality

$$(\sum_{i=1}^N a_i)^2 \le N \sum_{i=1}^N a_i^2$$

we have

$$
\begin{aligned}
&||\xi \circ F_k(t,\cdot) - \xi \circ F_k(s,\cdot)||_{L^2}^2 \\
&= ||\xi \circ f_h^{l-1} - \xi \circ f_h^{l'-1}||_{L^2}^2 \\
&\le (l'-l)\sum_{i=l-1}^{l'-2} ||\xi \circ f_h^i - \xi \circ f_h^{i+1}||_{L^2}^2 \\
&\le (l'-l)\sum_{i=l-1}^{l'-2} \frac{h}{2^{i+1}}(Dir(f_h^i;\mathbb{B}_1^m(0)) - Dir(f_h^{i+1};\mathbb{B}_1^m(0))) \\
&\le (l'-l)h\sum_{i=l-1}^{l'-2} (Dir(f_h^i;\mathbb{B}_1^m(0)) - Dir(f_h^{i+1};\mathbb{B}_1^m(0))) \\
&\le (l'-l)hDir(f_0;\mathbb{B}_1^m(0)) \le (s-t+h)Dir(f_0;\mathbb{B}_1^m(0)).
\end{aligned}
$$

The theorem is proved once we let $k \to \infty$ in the above inequality. □

# 4 Special cases of the flow

In this section, we will look at the case when $n = 1$, namely, $\mathbb{Q} = \mathbb{Q}_Q(\mathbb{R})$.

**Proposition 4.1.** *The minimizer $f_h^k$ satisfies the following equation:*

$$
\int_{\mathbb{B}_1^m(0)} < \phi, \frac{\eta(f_h^k) - \eta(f_h^{k-1})}{h/2^k} > + < D\phi, D(\eta(f_h^k)) > d\mathcal{L}^m = 0, \tag{34}
$$

*for any $\phi \in C_0^1(\mathbb{B}_1^m(0), \mathbb{R})$.*

*Proof.* For simplicity, denote

$$
f_h^k(x) = \sum_{i=1}^{Q}[[f_i(x)]], f_h^{k-1}(x) = \sum_{i=1}^{Q}[[f_i^0(x)]].
$$

Take any smooth function $\phi \in C_0^1(\mathbb{B}_1^m(0), \mathbb{R})$, let

$$
u_t(x) = f_h^k(x) + tQ[[\phi(x)]] = \sum_{i=1}^{Q}[[f_i(x) + t\phi(x)]] \in \mathcal{M}.
$$

From [AF]§2.3, we have

$$
\begin{aligned}
&Dir(u_t;\mathbb{B}_1^m(0)) \\
&= Dir(f_h^k;\mathbb{B}_1^m(0)) + 2Q\int_{\mathbb{B}_1^m(0)} < D(\eta \circ f_h^k), D(t\phi) > d\mathcal{L}^m + QDir(t\phi;\mathbb{B}_1^m(0)) \\
&= Dir(f_h^k;\mathbb{B}_1^m(0)) + 2tQ\int_{\mathbb{B}_1^m(0)} < D(\eta \circ f_h^k), D\phi > d\mathcal{L}^m + t^2QDir(\phi;\mathbb{B}_1^m(0))
\end{aligned} \tag{35}
$$

Fix any permutation $\sigma$ of $\{1, 2, \cdots, Q\}$,

$$\begin{aligned}
&\sum_{i=1}^{Q} |f_i(x) + t\phi(x) - f^0_{\sigma(i)}(x)|^2 \\
&= \sum_{i=1}^{Q} |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2|\phi(x)|^2 + 2t < \phi(x), f_i(x) - f^0_{\sigma(i)}(x) > \\
&= \sum_{i=1}^{Q} |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2 Q|\phi(x)|^2 + 2t < \phi(x), \sum_{i=1}^{Q} f_i(x) - \sum_{i=1}^{Q} f^0_{\sigma(i)}(x) > \\
&= \sum_{i=1}^{Q} |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2 Q|\phi(x)|^2 + 2t < \phi(x), Q(\eta \circ f^k_h - \eta \circ f^{k-1}_h) >
\end{aligned} \tag{36}$$

Therefore,

$$\mathcal{G}^2(u_t, f^{k-1}_h) = \mathcal{G}^2(f^k_h, f^{k-1}_h) + t^2 Q|\phi|^2 + 2tQ < \phi, \eta \circ f^k_h - \eta \circ f^{k-1}_h > \tag{37}$$

and

$$\begin{aligned}
G(f^{k-1}_h, u_t, h) &= Dir(u_t; \mathbb{B}^m_1(0)) + \int_{\mathbb{B}^m_1(0)} \mathcal{G}^2(u_t, f^{k-1}_h) d\mathcal{L}^m/(h/2^k) \\
&= Dir(f^k_h; \mathbb{B}^m_1(0)) + 2tQ \int_{\mathbb{B}^m_1(0)} < D(\eta \circ f^k_h), D\phi > d\mathcal{L}^m \\
&+ t^2 Q Dir(\phi; \mathbb{B}^m_1(0)) \\
&+ \frac{[\int_{\mathbb{B}^m_1(0)} \mathcal{G}^2(f^k_h, f^{k-1}_h) + t^2 Q|\phi|^2 + 2tQ < \phi, \eta \circ f^k_h - \eta \circ f^{k-1}_h > d\mathcal{L}^m]}{h/2^k}
\end{aligned} \tag{38}$$

Since $u_0 = f^k_h$ minimizes the functional $G(f^{k-1}_h, g, h)$,

$$\begin{aligned}
0 &= \frac{dG(f^{k-1}_h, u_t, h)}{dt}|_{t=0} \\
&= 2Q \int_{\mathbb{B}^m_1(0)} < D(\eta \circ f^k_h), D\phi > d\mathcal{L}^m + 2Q \int_{\mathbb{B}^m_1(0)} < \phi, \eta \circ f^k_h - \eta \circ f^{k-1}_h > d\mathcal{L}^m/(h/2^k),
\end{aligned} \tag{39}$$

which finishes the proof. □

From now on, $Q$ will be two, i.e, $\mathbb{Q} = \mathbb{Q}_2(\mathbb{R})$. We are going to look at a flow when the initial function is symmetric.
Given $g \in C^\infty(\mathbb{B}^m_1(0), \mathbb{R})$ such that $g$ is nonnegative, we define

$$f_0(x) = [[g(x)]] + [[-g(x)]] \in \mathcal{Y}_2(\mathbb{B}^m_1(0), \mathbb{Q}_2(\mathbb{R})).$$

Fix $h > 0$, let

$$f^0_h = f_0.$$

Therefore $\eta(f^0_h) \equiv 0$. By Proposition 4.1, $\eta(f^1_h)$ satisfies the following equality:

$$\int_{\mathbb{B}^m_1(0)} < \phi, \frac{\eta(f^1_h)}{h/2} > + < D\phi, D(\eta(f^1_h)) > d\mathcal{L}^m = 0, \tag{40}$$

for any $\phi \in C_0^1(\mathbb{B}_1^m(0), \mathbb{R})$.
It is easy to check that $\eta(f_h^1) \equiv 0$ (e.g [EL], §6.2). Similarly $\eta(f_h^k) \equiv 0, k = 1, 2, \cdots$.
In spirit of $\eta(f_h^1) \equiv 0$, we can write $f_h^1$ as

$$f_h^1 = [[f(x)]] + [[-f(x)]], f \geq 0.$$

Since $\xi \circ f_h^1 = (-f(x), f(x))$ and $\xi \circ f_h^1 \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{R}^2)$,

$$f \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{R}).$$

Take any nonnegative function $\phi \in C_0^1(\mathbb{B}_1^m(0), \mathbb{R})$, consider

$$f_t(x) = [[f(x) + t\phi(x)]] + [[-f(x) - t\phi(x)]] \in \mathcal{M}.$$

We have

$$\begin{aligned} Dir(f_t; \mathbb{B}_1^m(0)) &= 2\int_{\mathbb{B}_1^m(0)} |Df + tD\phi|^2 dx \\ &= 2\int_{\mathbb{B}_1^m(0)} [|Df|^2 + t^2|D\phi|^2 + 2tDf \cdot D\phi]dx \end{aligned} \tag{41}$$

and

$$\begin{aligned} \frac{1}{h/2}\int_{\mathbb{B}_1^m(0)} \mathcal{G}^2(f_h^0, f_t)dx &= \frac{4}{h}\int_{\mathbb{B}_1^m(0)} |f + t\phi - g|^2 dx \\ &= \frac{4}{h}\int_{\mathbb{B}_1^m(0)} [|f - g|^2 + t^2\phi^2 + 2t\phi(f - g)]dx \end{aligned} \tag{42}$$

Therefore,

$$0 = \frac{d}{dt}|_{t=0} G(f_h^0, f_t, h) = 4\int_{\mathbb{B}_1^m(0)} [Df \cdot D\phi + \frac{1}{h/2}\phi(f - g)]dx \tag{43}$$

for any nonnegative $\phi \in C_0^1(\mathbb{B}_1^m(0), \mathbb{R})$. Because of the linearity of the above equation, we conclude that $f$ is a weak solution of the following boundary-value problem:

$$\begin{cases} -\Delta u + \frac{u}{h/2} = \frac{g}{h/2} & \text{in } \mathbb{B}_1^m(0) \\ u = g & \text{on } \partial\mathbb{B}_1^m(0) \end{cases} \tag{44}$$

By introducing $\tilde{u} = f - g$, we see $\tilde{u}$ is a weak solution of the following boundary-value problem:

$$\begin{cases} -\Delta\tilde{u} + \frac{\tilde{u}}{h/2} = \Delta g & \text{in } \mathbb{B}_1^m(0) \\ \tilde{u} = 0 & \text{on } \partial\mathbb{B}_1^m(0) \end{cases} \tag{45}$$

which has a unique weak solution $\tilde{u}$. Moreover by the regularity theorem in [EL], §6.3,

$$\tilde{u} \in C^\infty(\mathbb{B}_1^m(0))$$

Hence $f \in C^\infty(\mathbb{B}_1^m(0))$ and $f$ is a smooth solution of the following PDE:

$$\frac{f - g}{h/2} = \Delta f.$$

Now let us denote:

$$f_h^k(x) = [[\tilde{f_h^k}(x)]] + [[-\tilde{f_h^k}(x)]], \tilde{f_h^k}(x) \geq 0, k = 0, 1, 2, \cdots.$$

Following the same argument, each $\tilde{f_h^k}$ is a smooth solution of the following PDE:

$$\frac{\tilde{f_h^k} - \tilde{f_h^{k-1}}}{h/2^k} = \Delta \tilde{f_h^k}.$$

This gives the proof of this theorem:

**Theorem 4.1.** *Suppose* $f_0 = [[g]] + [[-g]] \in \mathcal{Y}_2(\mathbb{B}_1^m(0), \mathbb{Q}_2(\mathbb{R}))$*, with nonnegative function* $g \in C^\infty(\mathbb{B}_1^m(0), \mathbb{R})$*. Then* $\eta(f_h^k) \equiv 0$ *for* $k = 0, 1, 2, \cdots$*. Moreover, if we denote*

$$f_h^k(x) = [[\tilde{f_h^k}(x)]] + [[-\tilde{f_h^k}(x)]], \tilde{f_h^k}(x) \geq 0, k = 0, 1, 2, \cdots.$$

*then each* $\tilde{f_h^k} \in C^\infty(\mathbb{B}_1^m(0), \mathbb{R})$ *and satisfies the following PDEs:*

$$\frac{\tilde{f_h^k} - \tilde{f_h^{k-1}}}{h/2^k} = \Delta \tilde{f_h^k}$$

Next, we will show that $f_h^k$ has no branch points. Namely,

**Theorem 4.2.** *If we have the same assumptions as the above theorem with the function g being not identically zero, then*

$$\tilde{f_h^k}(x) > 0, x \in \mathbb{U}_1^m(0), k = 1, 2, \cdots.$$

*In particular,* $f_h^k(x) \neq 2[[0]]$*, for any* $x \in \mathbb{U}_1^m(0)$*,* $k = 1, 2, \cdots$*.*

*Proof.* From Theorem 4.1 $\tilde{f_h^1}$ satisfies the following PDE:

$$\frac{\tilde{f_h^1} - \tilde{f_h^0}}{h/2} - \Delta \tilde{f_h^1} = 0.$$

Consider this operator:

$$Lu = -\Delta u + \frac{u}{h/2}. \tag{46}$$

Let $\psi = \tilde{f_h^1}$. We have

$$L\psi = \frac{\tilde{f_h^0}}{h/2} \geq 0. \tag{47}$$

By strong maximum principle of [EL], §6.4, we conclude that if $\psi$ attains a nonpositive minimum over $\mathbb{B}_1^m(0)$ at an interior point, then $\psi$ is constant within $\mathbb{B}_1^m(0)$. Since $\psi \equiv 0$ on $\partial \mathbb{B}_1^m(0)$, either $\psi > 0$ in $\mathbb{U}_1^m(0)$ or $\psi \equiv 0$ in $\mathbb{B}_1^m(0)$. But if $\psi \equiv 0$ in $\mathbb{B}_1^m(0)$, then $\tilde{f_h^0} = (h/2)L\psi \equiv 0$, which contradicts to our assumption about $g$. Therefore

$$\tilde{f_h^1} = \psi > 0 \text{ in } \mathbb{U}_1^m(0).$$

Now we assume that we have already showed that:

$$\tilde{f_h^k}(x) > 0, x \in \mathbb{U}_1^m(0).$$

From Theorem 4.1, $\tilde{f_h^{k+1}}$ satisfies the following PDE:

$$\frac{\tilde{f_h^{k+1}} - \tilde{f_h^k}}{h/2^{k+1}} - \Delta \tilde{f_h^{k+1}} = 0. \tag{48}$$

Consider the operator:

$$Lu = -\Delta u + \frac{u}{h/2^{k+1}}, \tag{49}$$

and let $\psi = \widetilde{f_h^{k+1}}$. Since

$$L\psi = \frac{\widetilde{f_h^k}}{h/2^{k+1}} > 0 \text{ in } \mathbb{U}_1^m(0), \tag{50}$$

the strong maximum principle tells either $\psi > 0$ in $\mathbb{U}_1^m(0)$ or $\psi \equiv 0$ in $\mathbb{B}_1^m(0)$.
$\psi$ cannot be identically zero because otherwise $\widetilde{f_h^k} = (h/2^{k+1})L\psi \equiv 0$, a contradiction to our induction assumption. Therefore, $\widetilde{f_h^{k+1}} = \psi > 0$ in $\mathbb{U}_1^m(0)$. □

**Remark 4.1.** *This in particular shows that the flow instantaneously separates, leaving no branch points.*

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