Theorem. Given a smooth function \( \tau(s) \) and a positive smooth function \( \kappa(s) \) on an interval \( I \) containing 0, a point \( \alpha_0 \in \mathbb{R}^3 \), and two unit vectors \( T_0 \) and \( N_0 \) in \( \mathbb{R}^3 \), there exists a unique unit-speed curve \( \alpha(s) \) on \( I \) with curvature \( \kappa(s) \), torsion \( \tau(s) \), initial position, \( \alpha(0) = \alpha_0 \), initial velocity \( \alpha'(s) = T_0 \), and initial acceleration \( \alpha''(s) = \kappa(0)N_0 \).

Proof. For the vector-valued function \( u \equiv (T,N,B) : I \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \), we may solve the O.D.E.

\[
\begin{align*}
T' &= \kappa N , \\
N' &= -\kappa T + \tau B , \\
B' &= -\tau N 
\end{align*}
\]

with initial conditions \( T(0) = T_0 \), \( N(0) = N_0 \), \( B(0) = T_0 \wedge N_0 \).

We claim that \( (T,N,B) \) is then automatically an orthonormal frame. To see this, note that vector-valued function

\[
v = (v_1, v_2, v_3, v_4, v_5, v_6) \equiv (T \cdot T, T \cdot N, T \cdot B, N \cdot N, N \cdot B, B \cdot B)
\]

then satisfies the O.D.E.

\[
\begin{align*}
v'_1 &= -2\kappa v_2 , \\
v'_2 &= -\kappa v_4 - \kappa v_1 + \tau v_3 , \\
v'_3 &= \kappa v_5 - \tau v_3 , \\
v'_4 &= -2\kappa v_2 + 2\tau v_6 \\
v'_5 &= -\kappa v_3 + \tau v_6 - \tau v_4 , \\
v'_6 &= -2\tau v_5
\end{align*}
\]

with \( v(0) = (1,0,0,1,0,1) \). But since the constant function \( (1,0,0,1,0,1) \) also satisfies the above O.D.E. with the same initial data, we conclude that \( v \equiv (1,0,0,1,0,1) \), so that \( (T,N,B) \) is indeed an orthonormal frame.

Also since the length of the vector function \( u = (T,N,B) \) remains bounded, in fact identically \( \sqrt{3} \), we see, by continuation, that the solution \( (T,N,B) \) exists not only near \( s = 0 \) but even over the whole interval \( I \).

We conclude that

\[
\alpha(s) = \alpha_0 + \int_0^s T(t) \, dt
\]

has \( \alpha(0) = \alpha_0 \) and is unit-speed with tangent \( T \) because \( \alpha'(s) = T(s) \). Also

\[
\alpha''(s) = T'(s) = \kappa N(s)
\]

so that \( \alpha \) has curvature \( \kappa \) and principal normal \( N \). Moreover,

\[
(T \wedge N)'(s) = (T' \wedge N)(s) + (T \wedge N')(s) = \tau(s)N(s)
\]
so that $\alpha$ also has torsion $\tau$.

Finally, if $\overline{\alpha}$ is another unit-speed curve with curvature $\kappa$ and torsion $\tau$, $\overline{\alpha}(0) = \alpha_0$, $\overline{\alpha}'(0) = T_0$, and $\overline{\alpha}''(0) = \kappa(0) N_0$, then the function $f(s) = T \cdot \overline{T} + N \cdot \overline{N} + B \cdot \overline{B}$ satisfies $f(0) = 3$ and, by the Frenet formulas, $f' \equiv 0$. Thus $f$ is the constant function 3, and each of the at most unit-sized terms $T \cdot \overline{T}$, $N \cdot \overline{N}$, $B \cdot \overline{B}$ must be the constant 1. Being unit vectors, $T \equiv \overline{T}$, $N \equiv \overline{N}$, $B \equiv \overline{B}$. In particular,

$$\overline{\alpha}(s) = \alpha_0 + \int_0^s \overline{T}(t) \, dt = \alpha_0 + \int_0^s T(t) \, dt = \alpha(s).$$

$\blacksquare$