1. Describe, with proof, four mutually non-isomorphic groups of order 50. In particular, construct the groups clearly and show carefully that the resulting groups are non-isomorphic.

2. Let $\Phi_{90}(x)$ denote the monic polynomial whose roots are the primitive 90th roots of unity; it is irreducible.
   a. Show that $\Phi_{90}(x) \in \mathbb{Z}[x]$, i.e., the coefficients are integers.
   b. Determine the splitting field of $\Phi_{90}$ as a polynomial over the finite field $\mathbb{F}_{11} = \mathbb{Z}/11\mathbb{Z}$.
   c. Now regard $\Phi_{90}$ as a polynomial over $\mathbb{Q}$. Describe, in detail, its Galois group.

3. Let $R$ be an integral domain. Assume that
   - $ab = cd$ holds, for some $a, b, c, d \in R$;
   - $a$ and $b$ are prime elements in $R$.

   **Prove or disprove:** The element $c$ must be an associate of one of the following elements: $a, b, ab, 1_R$ (the identity in $R$).

4. Let $A$ be a real $9 \times 9$ matrix with transpose $B$. Prove that the matrices $A$ and $B$ are real equivalent in the following sense: There exists a real
invertible $9 \times 9$ matrix $H$ such that $AH = HB$. For partial credit: Establish the existence of a complex invertible matrix $H$ with $AH = HB$.

5. Consider the rings

$$R := \mathbb{Z}[\sqrt{-3}] \subset S := \mathbb{Z}[\frac{1 + \sqrt{-3}}{2}] \subset \mathbb{C};$$

regard $S$ as an $R$ module.

a. Show that $S$ is finitely generated as an $R$-module.

b. Let $p \neq 0$ be a prime ideal of $R$ and consider the localizations

$$R_p \subset S_p.$$ 

Show these are equal if $p$ does not contain 2.

c. Show that $S$ is neither flat nor projective as an $R$ module.

6. Let $e_1, e_2, e_3, e_4$ be a basis for $\mathbb{R}^4$ and

$$q = e_1e_2 + e_3e_4 \in \text{Sym}^2(\mathbb{R}^4),$$ 

i.e., an element of the symmetric algebra $\text{Sym}(\mathbb{R}^4)$. Show there do not exist elements $v, w \in \mathbb{R}^4$ such that $q = vw$ in $\text{Sym}(\mathbb{R}^4)$. 

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