ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2018

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 16, 2018.

(1) Let p be a prime number and suppose that G and H are groups of order p^2. Suppose that G and H have the same number of subgroups of order p. Show that G and H are isomorphic.

(2) Let $G := \mathrm{GL}_3(\mathbb{F}_2)$ be the group of invertible 3×3 matrices over the field with two elements. Determine, up to similarity, all elements of G of order 7.

(3) Can $\sqrt{2+\sqrt{2}}$ be written in the form $\sqrt{a}+\sqrt{b}$ for some rational numbers a and b?

(4) Let $I = (x-y, y^2-y)$ and $J = (x^2y + xy^2 - 2y, xy^2 - x - y)$ be ideals in the polynomial ring $\mathbb{Q}[x,y]$. Is $I = J$? Justify.

(5) (a) Let $R = Z[\sqrt{-5}]$ and $I = (2, 1+\sqrt{-5}) \subset R$. Is $\bigwedge^2(I) = 0$? Justify.
 (b) Let $R = Z[x,y]$ and $I = (x,y) \subset R$. Is $\bigwedge^2(I) = 0$? Justify.

(6) Let R be a commutative ring with unit, and let M be an R-module. Prove that the following are equivalent:
 (a) $M = 0$.
 (b) $M_{\mathfrak{p}} = 0$ for all primes ideals $\mathfrak{p}$ of R.
 (c) $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R.