ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2018

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 9, 2018.

(1) Let S_n denote the symmetric group on n elements.

(a) Show that S_5 has **six** 5-Sylow subgroups.

(b) Prove that S_6 contains a subgroup isomorphic S_5 that acts transitively on the set $\{1, 2, 3, 4, 5, 6\}$.

(2) Let $\phi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism of abelian groups.

(a) Show that $\ker\phi = 0$ or is isomorphic to $\mathbb{Z}^k$ where $n - m \leq k \leq n$.

(b) For the homomorphism $\phi\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ given by

$$\phi\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 0 & -3 \\ 2 & 4 & 14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

show that the cokernel is a group of order 4. Is this group cyclic?

(3) Let $K = \mathbb{Q}(e^{2\pi i/5}) \subset \mathbb{C}$. Show that $K \cap \mathbb{Q}(i) = \mathbb{Q}$.

(4) Let V be a finite-dimensional complex vector space, and let $T\colon V \to V$ be a linear operator. Suppose that $T^n = \mathrm{Id}$ for some positive integer n.

(a) Let λ be an eigenvalue of T. Suppose that $v \neq 0$ satisfies $(T - \lambda\,\mathrm{Id})^2 v = 0$. Show that v is an eigenvector of T.

(b) Prove that T is diagonalizable, i.e., there exists a basis of V with respect to which the matrix of T is diagonal.

(5) Let V be an n-dimensional vector space over $\mathbb{Q}$, and let $T\colon V \to V$ be a linear operator. For each $m = 1, \ldots, n$, let

$$\bigwedge^m(f)\colon \bigwedge^m V \to \bigwedge^m V$$

denote the induced linear operator on the m-th exterior power of V.

(a) Assume that $n \geq 2$. Show that $\bigwedge^2(f) = 0$ if and only if the rank of f is ≤ 1.

(b) Show that $\bigwedge^n(f)$ is equal to scalar multiplication by the determinant $\det(f)$.

(6) Let R be a commutative ring with unit, and let M be a finitely generated R-module. Let $f\colon M \to M$ be a surjective homomorphism of R-modules. Prove that f is injective.