ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2019

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 7, 2019.

(1) Let p and q be primes numbers such that $p < q$ and $p \nmid q-1$. Prove that a group G of order pq is cyclic.

(2) Let A be an invertible $n \times n$ matrix over an arbitrary field k. Write A^t for the transpose of A and A^{-1} for the inverse of A.

   (a) Is A similar to A^t? If so, prove it; otherwise support your claim with a counterexample.

   (b) Is A^t similar to A^{-1}? If so, prove it; otherwise support your claim with a counterexample.

(3) Assume that for a positive integer m there exist infinitely many primes p such that

$$p \equiv 1 \bmod m.$$

Let G be a finite abelian group. Prove there exists a subfield K of a cyclotomic field such that $\mathrm{Gal}(K/\mathbb{Q}) \simeq G$.

(4) Let p be a prime number, and let $f(x) = x^4 + px + p \in \mathbb{Q}[x]$.

   (a) Suppose that $p \neq 3, 5$. Determine the Galois group of (the splitting field of) $f(x)$.

   (b) Now suppose that $p = 3$ or $p = 5$. Prove that the order of the Galois group of $f(x)$ divides 8.

(5) Let $\mathbb{F}_2$ denote a field with 2 elements, and let $\mathbb{F}_4$ be an extension of degree two of $\mathbb{F}_2$. Determine the set of prime ideals of the ring $\mathbb{F}_4 \otimes_{\mathbb{F}_2} \mathbb{F}_4$.

(6) Let M be a finitely generated module over a Noetherian ring A. Let $\phi\colon M \to M$ be an A-module endomorphism of M.

   (a) Prove that $\ker(\phi^n) \cap \mathrm{im}(\phi^n) = 0$ for all sufficiently large n.

   (b) Show that if ϕ is surjective, then ϕ is an isomorphism.