Analysis Exam, August 2018

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Let V be the linear subspace of $L^2([0,1], dx)$ spanned by the functions $1, x, x^2$. Determine the L^2 distance from the function x^3 to the subspace V.

2. Consider the function $f(z) = z^5 + 6z^3 - 10$, $z \in \mathbb{C}$. Find the number of zeros (counting multiplicity) of f in the region $2 < |z| < 3$.

3. Determine whether the iterated integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx\, dy$$

exists and, if so, compute its value. Justify every step.

4. Suppose that f and g are holomorphic on the punctured unit disk $0 < |z| < 1$.

   (a) If

$$\sup_{0<|z|<1} |z|^{1/3} |f(z)| < \infty,$$

   is the singularity 0 of f necessarily *removable*? (i.e. is f the restriction of a function holomorphic on the whole unit disk?). Explain your answer.

   (b) If

$$\sup_{0<|z|<1} |z|^{4/3} |g'(z)| < \infty,$$

   is the singularity 0 of g necessarily removable? Again explain your answer.

5. Let f be a measurable function on $\mathbb{R}$ such that $\displaystyle\int_{\mathbb{R}} |f(x)|\, dx < \infty$.

   (a) Find the limit $\displaystyle\lim_{y\to 0} \int_{\mathbb{R}} |f(x+y) - f(x)|\, dx$.

   (b) Find the limit $\displaystyle\lim_{y\to +\infty} \int_{\mathbb{R}} |f(x+y) - f(x)|\, dx$.

6. Let P be a complex polynomial of degree $d \geq 1$. Prove that the set

$$\{z \in \mathbb{C} \mid |P(z)| \neq 1\}$$

consists of at most $d+1$ connected components.