Analysis Exam, August 2019

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a differentiable function for $n = 1, 2, \ldots,$ that

$$M = \sup_{n,x} |f_n'(x)| < \infty,$$

and that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbb{R}$ exists for all $x \in \mathbb{R}$.

   (a) Is f continuous on $\mathbb{R}$? Prove or find a counterexample.

   (b) Is f differentiable on $\mathbb{R}$? Prove or find a counterexample.

   (c) Does $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$? Prove or find a counterexample.

2. How many zeros does $f(z) = z^4 - 6z + 3$ have in the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$?

3. Suppose f is analytic on $\mathbb{D} \setminus \{0\}$ and, for some $\epsilon > 0$, $\operatorname{Re} f(z) > 0$ whenever $0 < |z| < \epsilon$. Prove that f has a removable singularity at 0.

4. Prove that for every $f \in C([0,1])$,

$$\lim_{a\to\infty} a \int_0^1 e^{-ax} f(x)\,dx = f(0).$$

Hint: compute the limit for $f(x) = x^k$, $k = 0, 1, 2, \ldots$

5. Compute for $a \in (0,1)$ the value of the integral

$$\int_0^\infty \frac{t^a}{1+t^2}\,dt.$$

Hint: consider an analytic function on $\mathbb{C} \setminus [0,\infty)$ or a similar region.

6. (a) If $f_n \to f$ in $L^1([0,1], dx)$, prove that

$$\int_0^1 e^f\,dx \le \liminf_{n\to\infty} \int_0^1 e^{f_n}\,dx$$

   (b) Prove that strict inequality is possible.