Analysis Exam, January 2019

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. Determine whether the following limit exists and, if so, find its value:

$$\lim_{n\to\infty} \int_0^{1/n} \frac{n\,dt}{(1+n^2t^2)(1+t^2)}$$

2. Does there exist a nonconstant holomorphic function $f : \mathbb{C} \to \mathbb{C}$, satisfying $\lim_{|z|\to\infty} |z|^{-1/2}|f(z)| = 0$? If so, give an example. If not, explain why not.

3. Prove that, for all $f \in C([0,1])$,

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx \;=\; f(1)\ .$$

4. (a) Find a formula for a bijective map g between the upper half-plane $U = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ so that both g and g^{-1} are holomorphic.

   (b) Assume that f is an analytic function from U to itself such that $f(i) = i$. Prove that $|f'(i)| \leq 1$.

5. We say that a sequence of real numbers $(a_n)_{n=1}^{\infty}$ increases *subexponentially* if for every $\epsilon > 0$ there exists $M < \infty$ such that $|a_n| \leq Me^{\epsilon n}$ for all n.

   If $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, prove that the set

$$\{x \in \mathbb{R} \mid \text{the sequence } (f_n(x))_{n=1}^{\infty} \text{ increases subexponentially}\}$$

   is a Borel set.

6. (a) Find complex polynomials $p(z)$ and $q(z)$ so that

$$\cos\theta \;=\; \frac{p(e^{i\theta})}{q(e^{i\theta})} \quad \text{for all } \theta \in \mathbb{R}\ .$$

   (b) Compute, for any $a > 1$, the integral

$$\int_0^{\pi} \frac{1}{a+\cos\theta}\,d\theta.$$