Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Describe (with justification) a group of order 63 with 7 distinct 3-Sylow subgroups.

2. Let G_1, G_2, G_3 be three finite groups, with

$$|G_1| = |G_2| = m, \quad |G_3| = n.$$

   Assume that the product groups $G_1 \times G_3$ and $G_2 \times G_3$ are isomorphic.

   a. Prove that the groups G_1 and G_2 are also isomorphic if m, n are relatively prime.

   b. Prove that the groups G_1 and G_2 are also isomorphic if all groups are Abelian.

3. Find all complex numbers $\alpha \in \mathbb{C}$ such that the numbers α and $\beta = \alpha^2 + \alpha$ satisfy the same irreducible quadratic polynomial $p(x) \in \mathbb{Q}[x]$.

4. Let $\mathbb{F}_{11}$ denote the field with 11 elements, $p(x) = x^{13} + 1$, and $K/\mathbb{F}_{11}$ the splitting field of $p(x)$ over $\mathbb{F}_{11}$. What is $|K|$?

5. Consider the ring homomorphism $\phi : \mathbb{C}[x, y] \rightarrow \mathbb{C}[t]$, where $\phi(x) = t^2, \phi(y) = t^2 - t$, and $\phi(c) = c$ for each $c \in \mathbb{C}$. Show that the kernel $\ker(\phi)$ is a principal ideal.

6. Let M be a free abelian group of rank two and let $M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, i.e., group homomorphisms from M to $\mathbb{Z}$. An element $A \in M \otimes_{\mathbb{Z}} M^*$ is *decomposible* if there exist $m \in M$ and $m' \in M^*$ such that $A = m \otimes m'$.

   (a) Give an example of an $A \in M \otimes_{\mathbb{Z}} M^*$ that is not decomposible, justifying your answer.

   (b) Suppose that $A, B \in M \otimes_{\mathbb{Z}} M^*$ are not decomposible. Show there are at most finitely many $t \in \mathbb{Z}$ such that $A + tB$ is decomposible.