# Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. List all similarity classes of $3 \times 3$ complex matrices $C$ with the property that all the positive powers $C^n, n \geq 1$, are similar.

2. Let $K$ be the splitting field, over $\mathbb{Q}$, of $x^{12} - 1$. Describe the Galois group $G(K/\mathbb{Q})$ and describe all the subfields of $K$.

3. Let $p \in \mathbb{Z}$ be prime. Show that for any $n \geq 3$ there is non-abelian group of order $p^n$. Can any such group be simple? Justify your answer.

4. For $\alpha \in \mathbb{C}$ show that the following are equivalent:

   (i) $\alpha$ is an algebraic integer,

   (ii) $\mathbb{Z}[\alpha]$ (as an additive Abelian group) is finitely generated,

   (iii) $\mathbb{Z}[\alpha]$ (as an additive Abelian group) is free abelian of finite rank.

   Prove that the set of all algebraic integers in $\mathbb{C}$ is a ring.

5. Suppose that $R$ is a commutative integral domain and $Q$ is its field of fractions. Consider $Q$ as an $R$-module. Prove that $\bigwedge^2 Q = \{0\}$. (Here, the wedge product is over $R$.)

6. For parts $a - d$ below assume that $R = \mathbb{C}[x]$. Give a (non-zero) example or state "such example does not exist". Include brief justification. As part of your answers, define the concepts used in the questions (free, projective, injective).

   a. a free $R$-module that is finitely generated as a $\mathbb{C}$-module.

   b. an $R$-module that is finitely generated as a $\mathbb{C}$-module, that is not a projective $R$-module.

   c. a projective $R$-module that is not free.

   d. an injective $R$-module that is not free.