ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2016

Instructions:

- You have **four** hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 17th, 2016.

(1) (a) Prove that any group of order 1225 is abelian.

(b) Find all isomorphism classes of groups of order 1225.

(2) Determine the eigenvalues (with their algebraic multiplicity) of the following 8×8 matrix over the finite field $\mathbb{F}_5$

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \end{pmatrix}.$$

(3) Consider the polynomial in $\mathbb{F}_2[x]$

$$f(x) = x^5 + x^3 + 1.$$

(a) Show that $f(x)$ is irreducible.

(b) Let $K = \mathbb{F}_2[x]/(f(x))$ and set $w = x + (f(x))$. Express each of the roots of $f(x)$ in K as an $\mathbb{F}_2$-linear combination of $\{1, w, w^2, w^3, w^4\}$.

(c) Prove or Disprove: Any polynomial $g(x)$ in $\mathbb{F}_2[x]$ of degree five splits completely in K.

(4) Let p be a prime number, and let be n a positive integer.

(a) Let G be group of order p^n. Prove that a subgroup $H < G$ of index p is normal.

(b) Let L/K be a Galois extension of fields of degree p^n. Show there exists a Galois extension of K of degree p contained in L.

(5) Let $f\colon R \to S$ be a surjective homomorphism of rings (commutative, with unit). Let M and N be S-modules. Viewing M and N as R-modules via f, prove that $M \otimes_R N$ and $M \otimes_S N$ are isomorphic as R-modules.

(6) Let F be a field, $R \subset F$ be a subring, and let $\alpha \in F$ be a nonzero element such that $F = R[\alpha] = R[1/\alpha]$.

(a) Prove that F is integral over R.

(b) Prove that R is a field.