Algebra Qualifying Exam

Rice University Mathematics Department

May 4, 2009

You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Let M be an $n \times n$ matrix with coefficients in $\mathbb{R}$ such that $M^2 = -I$. Prove that n is even.

2. a. Show that every group of order 42 has a normal subgroup of index 6.

   b. Classify, up to isomorphism, groups of order 42 containing a subgroup isomorphic to S_3.

3. An idempotent in a ring is an element e such that $e^2 = e$. Prove:

   a. If e is an idempotent, then so is $1 - e$.

   b. The only idempotents in an integral domain are 0 and 1.

   c. The only idempotents in a local ring are 0 and 1.

   Give an example of a ring with idempotents different from 0 and 1.

4. Give an example of abelian groups X and Y such that

$$\mathrm{Ext}^1(X, Y) \neq \mathrm{Ext}^1(Y, X).$$

5. Let K be the splitting field of $x^4 + bx^2 + c \in \mathbb{Q}[x]$. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to a subgroup of the dihedral group with 8 elements.

6. Let A be an Artinian commutative ring and $f : A \to A$ an A-module homomorphism. Show that the following hold for sufficiently large integers k:

   a. $\mathrm{Im}(f^k) = \mathrm{Im}(f^{k+1})$.
   b. $\mathrm{Ker}(f^k) = \mathrm{Ker}(f^{k+1})$.
   c. $\mathrm{Im}(f^k) \cap \mathrm{Ker}(f^k) = 0$.
   d. $A = \mathrm{Im}(f^k) \oplus \mathrm{Ker}(f^k)$.