Algebra Qualifying Exam

Rice University Mathematics Department

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You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Let G be a finite group and denote by $S = S(G) = \{\mathrm{ord}(g) \mid g \in G\}$ the set of orders of the elements of G.

   a. Prove that if G is abelian then

   $$m, n \in S \Rightarrow \mathrm{l.c.m.}(m, n) \in S; \tag{1a}$$

   and that

   $$m \in S,\ n \in \mathbb{N} \text{ and } n \mid m \Rightarrow n \in S. \tag{1b}$$

   Here l.c.m. stands for the *least common multiple*, and the sign $\mid$ stands for *divides*.

   b. Suppose that G is a non-abelian group of odd order. Is (1a) still true? (Provide either a proof or a counterexample).

2. a. Prove that the polynomial $x^4 + nx + 1$ is irreducible over $\mathbb{Q}$, for every integer $n \neq \pm 2$.

   b. Determine whether the polynomial $x^2 + x + 1$ is irreducible over the finite field $\mathbb{F}_{101}$ with 101 elements.

3. In this problem, all rings are commutative domains with 1.

   a. Show that a unique factorization domain is integrally closed in its fraction field.

b. Show that $A := \mathbb{Q}[x,y]/\langle y^2 - x^3 \rangle$ is not a unique factorization domain.

c. Compute the integral closure of A in its fraction field.

4. Let V be an n-dimensional vector space over $\mathbb{Q}$, $f : V \to V$ a $\mathbb{Q}$-linear transformation. For each $m = 1, \ldots, n$ let

$$\bigwedge^m(f) : \bigwedge^m V \to \bigwedge^m V$$

denote the induced linear transformation on the mth exterior power.

a. Assume $n \geq 2$. Show that $\bigwedge^2(f) = 0$ if and only if $\operatorname{rank}(f) \leq 1$.

b. Show that $\bigwedge^n(f)$ is equal to scalar multiplication by the determinant $\det(f)$.

5. Consider the polynomial ideals

$$I = \langle b - r_1 - r_2, c - r_1 r_2 \rangle, J = I + \langle r_1 - r_2 \rangle \subset \mathbb{Q}[r_1, r_2, b, c],$$

where r_1, r_2, b, and c are indeterminants.

a. Show that $I \cap \mathbb{Q}[b,c] = \langle 0 \rangle$.

b. Compute $J \cap \mathbb{Q}[b,c]$ and interpret the generator(s).

6. Let A be a commutative ring with 1 and M an A-module.

a. Show that if M is a projective A-module then $\operatorname{Ext}^1_A(M,N) = 0$ for every A-module N.

b. Show that the following statements are equivalent:

(i) M is a flat A-module;

(ii) M_p is a flat A_p module for each prime ideal p of A.

(iii) M_m is a flat A_m module for each maximal ideal m of A.

Note that M_p and A_p denote the localizations of M and A at p.