Algebra Qualifying Exam, May, 2013

Time limit 3 hours. Use no books, notes, calculators, or other aids. Do as many problems as you can in any order. Concentrate on good exposition with complete and well reasoned arguments for all steps including calculations.

1. Prove that if G is a finite Abelian group, then the set

$$Orders(G) = \{o(g) : g \in G\}$$

(of orders of elements of G) is exactly the set of divisors of one integer $N = N(G)$.
Is this result true for all non-Abelian finite groups?

2. Let A be a commutative ring with unit, and let I and J be ideals of A. Show that

$$Tor_1^A(A/I, A/J) \cong (I \cap J)/IJ.$$

What else can you say if you also assume that $I + J = A$?

3. Let A be an invertible matrix over $\mathbb{C}$, and assume A is similar to some power A^n with $n \geq 2$. Show that there is an integer m such that $A^m - I$ is nilpotent.

4. Let A be a commutative ring with unit, let S be a multiplicative subset of A, and let M denote an A-module.
(a) Show that the map $f : S^{-1}A \otimes_A M \to S^{-1}M$ sending $a/s \otimes m$ to am/s (extended A-linearly), is an isomorphism of $S^{-1}A$-modules.
(b) Assume that in addition A is a domain. Show that $Frac(A)$ is flat over A.

5. Let F be a field of characteristic 0, For $i = 1, 2$ let $f_i(x) \in F[x]$, and K_i be its splitting field over F. Let K be the splitting field (over F) of $f_1(x)f_2(x)$. Show that the following are equivalent:
(i) $G(K/F) \cong G(K_1/F) \times G(K_2/F)$
(ii) $[K : F] = [K : K_1][K : K_2]$.
Does (i) follow just from the assumption that $f_1(x)$ and $f_2(x)$ are distinct and irreducible?

6. Let $I =< s - t^2 - x, st + y >$ be an ideal in the polynomial ring $\mathbb{Q}[s, t, x, y]$.
(a) Compute a reduced Groebner basis for I with respect to the lexicographic order $s > t > x > y$.
(b) Show that the polynomial ring homomorphism $\mathbb{Q}[x, y] \to \mathbb{Q}[x, y]$ determined by $x \mapsto x - y^2$ and $y \mapsto -xy$ is injective.

PLEDGE