ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2014

Instructions:

- You have 4 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

(1) Classify, up to isomorphism, groups of order 21.

(2) Let $S = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$, and let $\mathrm{GL}_4(\mathbb{Q})$ denote the group of invertible 4×4 matrices with rational entries. Write Id for the identity of the group $\mathrm{GL}_4(\mathbb{Q})$.

  (a) Show that if $A \in \mathrm{GL}_4(\mathbb{Q})$ has order n, then n is an element of the set S. [Recall that the order of a matrix A is the smallest positive integer n satisfying $A^n = \mathrm{Id}$.]

  (b) For each of the nine elements $n \in S$, write down a matrix $A \in \mathrm{GL}_4(\mathbb{Q})$ of order n.

(3) Let $\mathbb{F}$ be a finite field, and write $\mathbb{Q}$ for the field of rational numbers.

  (a) Which groups of order 6 are Galois groups of polynomials with coefficients in $\mathbb{F}$?

  (b) For each group G of order 6, determine a polynomial $f_G(x) \in \mathbb{Q}[x]$ with Galois group isomorphic to G and find all the subfields of the splitting field of f_G.

(4) Let $\mathbb{F}_2$ denote a field with 2 elements, and let $\mathbb{F}_4$ be an extension of degree two of $\mathbb{F}_2$. Determine the set of prime ideals of the ring $\mathbb{F}_4 \otimes_{\mathbb{F}_2} \mathbb{F}_4$.

(5) (a) Write down polynomial equations for the image of the circle

$$C = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

  under the mapping

$$(x, y) \mapsto (xy, x^2 - y^2).$$

  (b) Let $f(x, y) \in \mathbb{R}[x, y]$ be a polynomial such that

$$f(x, y), x^2 + y^2 - 1\rangle = \mathbb{R}[x, y].$$

  Show that for each point $p \in C$, we have $f(p) \neq 0$.

  (c) Let $f(x, y) \in \mathbb{R}[x, y]$ be a polynomial such that f has no zeros on C. Does it follow that

$$f(x, y), x^2 + y^2 - 1\rangle = \mathbb{R}[x, y]?$$

(6) Let $R = \mathbb{C}[x, y]$, $I = \langle x, y\rangle$, and $Q = R/I$. For each nonnegative integer i, compute the groups $\mathrm{Tor}_i^R(R, Q)$ and $\mathrm{Tor}_i^R(Q, Q)$. State explicitly which of these groups are zero.

Date: May 14th, 2014.