ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2016

January 12, 2016

Instructions:

(1) You have 4 hours to complete this exam. Attempt all **six** problems
(2) The use of books, notes, calculators, or other aids is **not** permitted.
(3) Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
(4) Write and sign the Honor Code pledge at the end of your exam.

Problem 1. A group G is called *special* if for any pair $x, y \in G$ of its elements either $y \in \langle x \rangle$ or $x \in \langle y \rangle$ (or both). (Here $\langle x \rangle$ stands for the subgroup of G generated by x).

(1A) List all special groups of orders ≤ 30. (full credit)
(1B) Do there exist infinite special groups? (extra credit)

Problem 2. Find the number of non-isomorphic finite fields F of orders at most 50 which have a non-trivial 7th root of 1_F. (That is, the equation $x^7 = 1_F$ has a root $x \in F,\ x \neq 1_F$).

Problem 3. Let $K \subset \mathbb{C}$ be a subfield such that $1 \leq \big[K : \mathbb{Q}\big] = n < \infty$. Denote

$$K' = \{\bar{k} \mid k \in K\} \subset \mathbb{C}, \qquad K_2 = \Big\{|k|^2 \;\Big|\; k \in K \quad \subset \mathbb{R}$$

(where $\bar{k}$ stands for the complex conjugate of k).

(3A) Provide an example of such K with $K \neq K'$.
(3B) Prove or disprove: K' and K must be isomorphic fields.
(3C) Prove that $m = \big[\mathbb{Q}(K_2) : \mathbb{Q}\big]$ must be finite where $\mathbb{Q}(K_2)$ denotes the field generated by the set K_2. Find an upper estimate on m in terms of $n = \big[K : \mathbb{Q}\big]$.

Problem 4. Prove that the polynomial $x^4 + nx + 1$ is irreducible over $\mathbb{Q}$ for every integer $n \neq \pm 2$.

Problem 5. Denote by M the set of 7×7 matrices over $\mathbb{C}$. Find the number of non-equivalent nilpotent matrices in M of rank 5. (Two matrices $A, B \in M$ are called equivalent if there exists an invertible matrix $C \in M$ such that $AC = CB$).

Problem 6. Let $\phi\colon R \to S$ be a surjective homomorphism of commutative rings with unity. Prove that $\phi(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(S)$ where $\mathrm{Jac}(\cdot)$ is the Jacobson radical (the intersection of all maximal ideals of the corresponding ring).