Algebra Qualifying Exam

Rice University Mathematics Department

January 4, 2009

You have three hours to complete this exam. Please use no books, notes, calculators, or other aids. Remember to complete the Honor Code pledge with your exam. Please give arguments for all your answers, including computations!

1. Let p be an odd prime number.

   a. Let G be a group of order $2p$. Show that G is a semidirect product of cyclic groups $C_2 \ltimes C_p$.

   b. Show that for each p, there exist precisely two groups of order $2p$, up to isomorphism.

2. Consider the polynomial $f(x) = x^{11} + 1$. Describe the number and degrees of the irreducible factors of $f(x)$ in $k[x]$ over the following fields:

   a. $k = \mathbb{Q}$

   b. $k = \mathbb{F}_7$, the finite field with seven elements;

   c. $k = \mathbb{F}_{2^e}$ for $e \in \mathbb{N}$.

   In case (a), compute the Galois group of the splitting field of f over k.

3. a. Let k be a field and $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ be polynomials such that there exist $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ with

   $$f_1 g_1 + \ldots + f_r g_r = 1.$$

   Show that $f_1, \ldots, f_r$ have no common zeros $(a_1, \ldots, a_n) \in k^n$.

b. Now let $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials with no common complex zeros. Show there exist $g_1, \ldots, g_r \in \mathbb{C}[x_1, \ldots, x_n]$ with

$$f_1 g_1 + \ldots + f_r g_r = 1.$$

c. Suppose that $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n]$ are polynomials with no common *real* zeros. Do there exist $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$ with

$$f_1 g_1 + \ldots + f_r g_r = 1?$$

Give a proof or counterexample.

4. Let $G = \mathrm{SL}_2(\mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant one. Suppose that $A \in G$ has finite order, i.e., $A^N = I$ for some $N \in \mathbb{N}$; let N_0 be the smallest such N.

   a. Give examples where $N_0 = 1, 2, 3$, and 6.

   b. Show that $N_0 = 1, 2, 3, 4$, or 6.

5. Let R be a commutative ring with 1, $S \subset R$ a multiplicative subset, and $R[S^{-1}]$ the corresponding localization.

   a. Show that $R[S^{-1}]$ is Noetherian if R is Noetherian.

   b. Show that $R[S^{-1}]$ is a principal ideal domain if R is a principal ideal domain.

6. Let a and b be distinct rational numbers, and m and n positive integers. Write $R = \mathbb{Q}[x]$ and compute the following:

   a. $\mathrm{Hom}_R(R/\langle (x-a)^m \rangle, R/\langle (x-b)^n \rangle)$;

   b. $(R/\langle (x-a)^m \rangle) \quad {}_R\, (R/\langle (x-b)^m \rangle)$;

   c. $\dim_{\mathbb{Q}}(R/\langle (x-a)^m \rangle) \quad {}_{\mathbb{Q}}\, (R/\langle (x-b)^m \rangle)$, where the quotient rings are regarded as vector spaces over $\mathbb{Q}$;

   d. $\mathrm{Tor}_1^R(R/\langle (x-a)^m \rangle, R/\langle (x-b)^n \rangle)$.