ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2015

Instructions:

- You have 3 hours to complete this exam. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 14th, 2015.

(1) Let p be a prime integer. Let G be a finite group, H a normal subgroup of G, and K a Sylow p-subgroup of G. Prove that $K \cap H$ is a Sylow p-subgroup of H.

(2) Consider the polynomial

$$f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

Determine whether it is irreducible over each of the following fields:

(a) $\mathbb{Z}/11\mathbb{Z}$.

(b) $\mathbb{Z}/19\mathbb{Z}$.

(c) The field with 361 elements.

(3) Compute the Galois groups of the splitting fields for the following polynomials over $\mathbb{Q}$. Carefully justify your answers:

(a) $x^3 + 4x + 2$.

(b) $x^4 - 5x^2 + 6$.

(4) Fix an integer $n \geq 1$, a field K and an algebraic closure $\overline{K}$ of K. Let A be an $n \times n$ matrix over K. Assume that all the roots of the characteristic polynomial of A are distinct as elements of $\overline{K}$. Let B and C be $n \times n$ matrices over K that commute with A. Prove that B and C commute.

(5) Let R be a commutative ring with unit. Suppose that $R \neq 0$. Prove that if $R^m \cong R^n$ then $m = n$.

(6) Let p and q be distinct prime numbers. Compute the abelian groups

$$\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/q\mathbb{Z})$$

for all $n \geq 0$.