Analysis Exam, August 2012

1. (a) Suppose a is a point in the open unit disk D in $\mathbf{C}$. Find a formula for a bijective holomorphic map $T_a : D \to D$ such that $T_a(a) = 0$.

(b) Verify that your T_a does satisfy the bijective property.

(c) Is your T_a the only holomorphic bijection of D which vanishes at a? If so, explain why. If not, find the *general* formula.

2. Assume that $f \in L^1([0,1])$. For each of the following, decide whether it must necessarily also belong to $L^1([0,1])$. It it does, explain why. If not, give a specific counterexample.

(a) $\sqrt{|f|}$.

(b) f^2 .

(c) Arctan f .

3. Compute

$$\int_{-\infty}^{0} \frac{x^{1/3}}{x^5 - 1}\, dx \; .$$

4. (a) Find continuous functions f_n on $[0,1]$ such that $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in [0,1]$, but $\int_0^1 f_n(t)\, dt$ does not have limit 0 as $n \to \infty$.

(b) Find continuous functions g_n on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 |g_n(t)|\, dt = 0$, but for *no point* $t \in [0,1]$ is it true that $\lim_{n\to\infty} g_n(t) = 0$.

(c) Find continuous functions h_n on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 h_n(t)\phi(t)\, dt = 0$ for all continuous functions ϕ on $[0,1]$, but for *no point* $t \in [0,1]$ is it true that $\lim_{n\to\infty} h_n(t) = 0$.

5. (a) Suppose $0 < r < 1$, $A = \{z \in \mathbf{C} : r < |z| < 1\}$, $f : \overline{A} \to \mathbf{C}$ is continuous, f is holomorphic on A, and f vanishes on the unit circle. What can you conclude about f? Prove your answer.

(b) Suppose g is holomorphic on the open unit disk D. Prove there exist points $a_n \in D$, $a \in \partial D$ and $\lambda \in \mathbf{C}$ so that $a_n \to a$ and $g(a_n) \to \lambda$ as $n \to \infty$.

6. (a) Show that, for $1 \le p \le \infty$, there exists a constant c_p so that $\|f\|_{L^1} \le c_p \|f\|_{L^p}$ for all $f \in L^p([-1,1])$.

(b) Show that there exists a constant c so that so that $\|g\|_{L^1} \le c\big(\|g\|_{L^2} + \|xg\|_{L^2}\big)$ whenever $g,\ xg \in L^2(\mathbf{R})$.