Qualifying Examination
Analysis

Thursday, August 20, 2015

9:00AM-1:00PM

There are six problems.

To ensure adequate space for your answer, there is

a blank page following the page on which a problem occurs.

Please **PRINT** your name clearly on this page.

"Ahlfors" means Complex Analysis, Third Edition.

"Royden" means Real Analysis, Third Ediition.

Problem 1 *Let $\{a_n | n \geq 1\}$ be the Fibonacci sequence:*

$$a_1 = a_2 = 1$$

$$a_{n+1} = a_n + a_{n-1}, n \geq 2.$$

A. Prove that the power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

has a positive radius of convergence. A simple induction shows that $0 < a_n < 2^n, n > 0$. Therefore, the radius is at least $\frac{1}{2}$. See Ahlfors, p. 38.

B. Prove that f is a rational function, and give a formula for it.

$$f(z) = z + 2z^2 + \sum_{n=3}^{\infty} a_n z^n$$

$$= z + 2z^2 + \sum_{n=3}^{\infty} (a_{n-2} + a_{n-1}) z^n = z + 2z^2 + z^2 \sum_{k=1}^{\infty} a_k z^k + z \sum_{l=2}^{\infty} a_l z^l$$

$$= z + z^2 + z^2 \sum_{k=1}^{\infty} a_k z^k + z \sum_{l=1}^{\infty} a_l z^l = (z + z^2)(1 + f(z)).$$

$$f(z) = \frac{z + z^2}{1 - z - z^2}.$$

C. State a theorem that allows you to identify the radius of convergence of the power series (1) without further computation. $1 - z - z^2$ has $\frac{\sqrt{5}-1}{2}$ for its smallest root. A holomorphic function is represented by its Taylor series about zero on any disc centered at zero in its domain. Ahlfors, p. 179.

Problem 2 *1. Let f be* **bounded** *and holomorphic on the punctured disc, $\Delta_0 = \{z|0 < |z| < 1\}$. State a theorem concerning the possibility of assigning a value to f at $z = 0$ so that the extended function is holomorphic on the disc $\Delta = \{z|\ |z| < 1\}$.*

2. Prove the theorem stated in 1.

Ahlfors, pp 124 ff. for Parts 1 and 2.

3. Is the same statement true if the function is real-valued and the word 'harmonic' is used in place of the word 'holomorphic'? Explain your answer.

See Ahlfors, pp. 166-168 for Poisson's formula. Fix $R > 0$ so that the real function f is bounded and harmonic on the disc of radius $2R$ centered at zero. Use the Poisson formula to construct an harmonic function, F, on the disc $\Delta = \{z|0 < |z| \leq R\}$ with $F|_{|z|=R} = f|_{|z|=R}$. The function $g = F - f$ is continuous on $\Delta \backslash \{0\}$, harmonic for $0 < |z| < R$ and it vanishes on $|z| = R$. We claim that g vanishes identically on $\Delta \backslash \{0\}$, so that F provides the harmonic continuation of f. It is sufficient to prove for any fixed $z \in \Delta$ that $g(z) \leq 0$ and $g(z) \geq 0$. It is given that $M = \|f\|_\infty < \infty$ and therefore, e.g., by the maximum principle for F, on Δ, $\|g\|_\infty < 2M$. Let $\varepsilon > 0$ be arbitrary. For all sufficiently small values of $r > 0$ it is true that $|\varepsilon \log r| > 2M$. Therefore, by the maximum (or minimum) principle, the functions $g_\pm = g \pm \varepsilon \log|z|, r < |z| < R$ satisfy

$$g_+(w) < 0 < g_-(w), r < |w| < R$$

and therefore, (1)

$$\varepsilon \log|w| < g(w) < -\varepsilon \log|w|, r < |w| < R.$$

Let $z \in \Delta$ and $\varepsilon > 0$ be given. Choose r as above, and then decrease it, if necessary, to guarantee that $r < |z|$. From (1)

$$\varepsilon \log|z| < g(z) < -\varepsilon \log|z|.$$

Since z is fixed and ε is arbitrarily small, it follows that

$$0 \leq g(z) \leq 0.$$

Problem 3 *Is it possible to define a single-valued holomorphic branch of* $(z(1-z))^{\frac{1}{2}}$ *on* $\mathbb{C}\setminus[0,1]$*? Explain why or why not.*

If γ *is a smooth closed curve in* $\mathbb{C}\setminus[0,1]$ *the function* $N(\gamma,x)$*,*

$$N(\gamma,x)=\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-x},0\leq x\leq 1,$$

is continuous and integer-valued on the unit interval, hence constant. In particular,

$$\frac{1}{2\pi i}\int_{\gamma}\left(\frac{1}{z}-\frac{1}{z-1}\right)dz=\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z(1-z)}=0.$$

This means that for any $z_0\in\mathbb{C}\setminus[0,1]$ *the line integrals*

$$\varphi(z)=w_0+\int_{z_0}^{z}\frac{d\zeta}{\zeta(1-\zeta)}$$

are independent of path in $\mathbb{C}\setminus[0,1]$*. The function* φ *is holomorphic on* $\mathbb{C}\setminus[0,1]$ *with derivative* $\frac{1}{z(1-z)}$*, the same derivative as any single-valued branch of* $\log z(1-z)$ *on a region in* $\mathbb{C}\setminus[0,1]$*. In particular, if* $e^{w_0}=z_0$*, then* φ *is a single-valued branch of* $\log z(1-z)$ *on* $\mathbb{C}\setminus[0,1]$*. Since*

$$e^{\varphi(z)}=z(1-z),z\in\mathbb{C}\setminus[0,1],$$

the function $\psi(z)=e^{\frac{1}{2}\varphi(z)}$ *is an holomorphic square root of* $(z(1-z))^{\frac{1}{2}}$ *on* $\mathbb{C}\setminus[0,1]$*.*

NOTATIONS and ASSUMPTIONS. Denote the Lebesgue measure by dx. $F(x)$ is a differentiable function on $\mathbb{R}$ (not necessarily **continuously** differentiable). The derivative, $F'(x)$, is assumed to be locally bounded on $\mathbb{R}$. That is, for any $T < \infty$,

$$-M \leq F'(x) \leq M \text{ for all } x \in [-T, T], \text{ where } M = M(T) < \infty.$$

Problem 4 *1. Prove that the derivative, $F'(x)$, is Borel measurable.*

This is true because a differentiable function is continuous, the derivative satisfies

$$\lim_{n \to \infty} \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}} = F'(x), x \in \mathbb{R},$$

and the pointwise limit of a pointwise convergent sequence of Borel functions is Borel.

2. *Prove that the Lebesgue integral of $F'(x)$ over any finite interval satisfies*

$$\int_a^b F'(x)\, dx = F(b) - F(a), -\infty < a < b < \infty.$$

Lebesgue's bounded convergence theorem implies

$$\lim_{n \to \infty} \int_a^b \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}} dx = \int_{[a,b]} F'(x) dx.$$

On the other hand, for each sufficiently large $n > 0$, it is true that

$$\int_a^b \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}} dx = n \int_{b - \frac{1}{n}}^{b} F\left(x + \frac{1}{n}\right) dx - n \int_a^{a + \frac{1}{n}} F(x)\, dx$$

Continuity of F, cited above, implies the right side converges to $F(b) - F(a)$.

Problem 5 *Let f and g be real-valued, Lebesgue measurable functions on $\mathbb{R}$. Prove that $\{x | f(x) = g(x)\}$ is a Lebesgue measurable set. Your proof should use the definition of measurability. It is not allowed to say simply that "$f - g$ is measurable, therefore...."*

It is sufficient to prove $\{x | f(x) > g(x)\}$ and $\{x | f(x) < g(x)\}$ are measurable. Clearly,

$$\{x | f(x) > g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x | g(x) < r < f(x)\}$$

$$= \bigcup_{r \in \mathbb{Q}} (\{x | g(x) < r\} \cap \{x | r < f(x)\}),$$

which is a countable union of measurable sets. A similar argument is used for $\{x | f(x) < g(x)\}$. The set in question is the complement of the unions, hence measurable.

Problem 6 *Prove directly, i.e., without the help of Stirling's formula, that*

$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}.$$

Represent the logarithm of $\frac{(n!)^{\frac{1}{n}}}{n}$ *in terms of the sums*

$$\log\left(\frac{(n!)^{\frac{1}{n}}}{n}\right) = \tfrac{1}{n}\log n! - \log n$$

$$= \tfrac{1}{n}(\log n! - n\log n) = \tfrac{1}{n}\sum_{k=0}^{n-1}\log\left(1-\tfrac{k}{n}\right).$$

Define step functions, $f_n(x)$, *by*

$$f_n(x) = \sum_{k=0}^{n-1}\left(\log\left(1-\frac{k}{n}\right)\right)\chi_{[\frac{k}{n},\frac{k+1}{n})}(x).$$

Observe that the sequence $\{f_n(x)\}$ *is dominated by the function* $|\log(1-x)|$, *which is easily checked to be Lebesgue integrable,*

$$|f_n(x)| \le |\log(1-x)|, 0 \le x \le 1.$$

The dominated convergence theorem implies, using $(t\log t - t)' = \log t, 0 < t$,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\tfrac{1}{n}\sum_{k=1}^{n-1}\log\left(1-\tfrac{k}{n}\right)$$

$$= \int_0^1 \log(1-x)\,dx = \int_0^1 \log y\;dy$$

$$= \lim_{t\searrow 0}\int_t^1 \log y\;dy = \lim_{t\searrow 0}(t\log t - t)\,|_t^1 = -1.$$

The substitution $y = 1-x$ *does not change the Lebesgue integral because Lebesgue measure is preserved by the substitution. Exponentiate to obtain the desired result.*